XXIX. *An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels.*

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Received and Read March 15, 1883.

[PLATES 72–74.]

SECTION I.

Introductory.

1. *Objects and results of the investigation.*—The results of this investigation have both a practical and a philosophical aspect.

In their practical aspect they relate to the *law of resistance to the motion of water in pipes,* which appears in a new form, the law for all velocities and all diameters being represented by an equation of two terms.

In their philosophical aspect these results relate to the fundamental principles of fluid motion; inasmuch as they afford for the case of pipes a definite verification of two principles, which are—*that the general character of the motion of fluids in contact with solid surfaces depends on the relation between a physical constant of the fluid and the product of the linear dimensions of the space occupied by the fluid and the velocity.*

The results as viewed in their philosophical aspect were the primary object of the investigation.

As regards the practical aspect of the results it is not necessary to say anything by way of introduction; but in order to render the philosophical scope and purpose of the investigation intelligible it is necessary to describe shortly the line of reasoning which determined the order of investigation.

2. *The leading features of the motion of actual fluids.*—Although in most ways the exact manner in which water moves is difficult to perceive and still more difficult to define, as are also the forces attending such motion, certain general features both of the forces and motions stand prominently forth, as if to invite or to defy theoretical treatment.

The relations between the resistance encountered by, and the velocity of, a solid body moving steadily through a fluid in which it is completely immersed, or of water moving through a tube, present themselves mostly in one or other of two simple forms. The resistance is generally proportional to the square of the velocity, and when this is not the case it takes a simpler form and is proportional to the velocity.

Again, the internal motion of water assumes one or other of two broadly distinguishable forms—either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination, or they eddy about in sinuous paths the most indirect possible.

The transparency or the uniform opacity of most fluids renders it impossible to see the internal motion, so that, broadly distinct as are the two classes (direct and sinuous) of motion, their existence would not have been perceived were it not that the surface of water, where otherwise undisturbed, indicates the nature of the motion beneath. A clear surface of moving water has two appearances, the one like that of *plate glass*, in which objects are reflected without distortion, the other like that of *sheet glass*, in which the reflected objects appear crumpled up and grimacing. These two characters of surface correspond to the two characters of motion. This may be shown by adding a few streaks of highly coloured water to the clear moving water. Then although the coloured streaks may at first be irregular, they will, if there are no eddies, soon be drawn out into even colour bands; whereas if there are eddies they will be curled and whirled about in the manner so familiar with smoke.

3. *Connexion between the leading features of fluid motion.*—These leading features of fluid motion are well known and are supposed to be more or less connected, but it does not appear that hitherto any very determined efforts have been made to trace a definite connexion between them, or to trace the characteristics of the circumstances under which they are generally presented. Certain circumstances have been definitely associated with the particular laws of force. Resistance, as the square of the velocity, is associated with motion in tubes of more than capillary dimensions, and with the motion of bodies through the water at more than insensibly small velocities, while resistance as the velocity is associated with capillary tubes and small velocities.

The equations of hydrodynamics, although they are applicable to *direct motion, i.e.*, without eddies, and show that then the resistance is as the velocity, have hitherto thrown no light on the circumstances on which such motion depends. And although of late years these equations have been applied to the theory of the eddy, they have not been in the least applied to the motion of water which is a mass of eddies, *i.e.*, in *sinuous motion*, nor have they yielded a clue to the cause of resistance varying as the square of the velocity. Thus, while as applied to waves and the motion of water in capillary tubes the theoretical results agree with the experimental, the theory of hydrodynamics has so far failed to afford the slightest hint why it should explain these phenomena, and signally fail to explain the law of resistance encountered by large bodies moving at sensibly high velocities through water, or that of water in sensibly large pipes.

This accidental fitness of the theory to explain certain phenomena while entirely failing to explain others, affords strong presumption that there are some fundamental principles of fluid motion of which due account has not been taken in the theory. And several years ago it seemed to me that a careful examination as to the connexion between these four leading features, together with the circumstances on which they severally depend, was the most likely means of finding the clue to the principles overlooked.

4. *Space and velocity.*—The definite association of resistance as the square of the velocity with sensibly large tubes and high velocities, and of resistance as the velocity with capillary tubes and slow velocities seemed to be evidence of the very general and important influence of some properties of fluids not recognised in the theory of hydrodynamics.

As there is no such thing as absolute space or absolute time recognised in mechanical philosophy, to suppose that the character of motion of fluids in any way depended on absolute size or absolute velocity, would be to suppose such motion without the pale of the laws of motion. If then fluids in their motions are subject to these laws, what appears to be the dependance of the character of the motion on the absolute size of the tube and on the absolute velocity of the immersed body, must in reality be a dependance on the size of the tube as compared with the size of some other object, and on the velocity of the body as compared with some other velocity. What is the standard object and what the standard velocity which come into comparison with the size of the tube and the velocity of an immersed body, are questions to which the answers were not obvious. Answers, however, were found in the discovery of a circumstance on which sinuous motion depends.

5. *The effect of viscosity on the character of fluid motion.*—The small evidence which clear water shows as to the existences of internal eddies, not less than the difficulty of estimating the viscous nature of the fluid, appears to have hitherto obscured the very important circumstance that *the more viscous a fluid is, the less prone is it to eddying or sinuous motion.* To express this definitely—if μ is the viscosity and ρ the density of the fluid—for water $\frac{\mu}{\rho}$ diminishes rapidly as the temperature rises, thus at 5° C. $\frac{\mu}{\rho}$ is double what it is at 45° C. What I observed was that the tendency of water to eddy becomes much greater as the temperature rises.

Hence connecting the change in the law of resistance with the birth and development of eddies, this discovery limited further search for the standard distance and standard velocity to the physical properties of the fluid. To follow the line of this search would be to enter upon a molecular theory of liquids, and this is beyond my present purpose. It is sufficient here to notice the well known fact that

$$\frac{\mu}{\rho} \text{ or } \mu'$$

is a quantity of the nature of the product of a distance and a velocity.

It is always difficult to trace the dependance of one idea on another. But it may be noticed that no idea of dimensional properties as indicated by the dependance of the character of motion on the size of the tube and the velocity of the fluid, occurred to me until after the completion of my investigation on the transpiration of gases, in which was established the dependance of the law of transpiration on the relation between the size of the channel and the *mean range* of the gaseous molecules.

6. *Evidence of dimensional properties in the equations of motion.*—The equations of motion had been subjected to such close scrutiny, particularly by Professor STOKES, that there was small chance of discovering anything new or faulty in them. It seemed to me possible, however, that they might contain evidence which had been overlooked, of the dependance of the character of motion on a relation between the dimensional properties and the external circumstances of motion. Such evidence, not only of a connexion but of a definite connexion, was found, and this without integration.

If the motion be supposed to depend on a single velocity parameter U, say the mean velocity along a tube, and on a single linear parameter c, say the radius of the tube; then having in the usual manner eliminated the pressure from the equations, the accelerations are expressed in terms of two distinct types. In one of which

$$\frac{U^2}{c^3}$$

is a factor, and in the other

$$\frac{\mu U}{\rho c^4}$$

is a factor. So that the relative values of these terms vary respectively as U and

$$\frac{\mu}{c\rho}.$$

This is a definite relation of the exact kind for which I was in search. Of course without integration the equations only gave the relation without showing at all in what way the motion might depend upon it.

It seemed, however, to be certain if the eddies were owing to one particular cause, that integration would show the birth of eddies to depend on some definite value of

$$\frac{c\rho U}{\mu}$$

7. *The cause of eddies.*—There appeared to be two possible causes for the change of direct motion into sinuous. These are best discussed in the language of hydrodynamics, but as the results of this investigation relate to both these causes, which, although the distinction is subtle, are fundamentally distinct and lead to distinct results, it is necessary that they should be indicated.

The general cause of the change from steady to eddying motion was in 1843 pointed out by Professor STOKES as being that under certain circumstances the steady motion becomes unstable, so that an indefinitely small disturbance may lead to a change to sinuous motion. Both the causes above referred to are of this kind, and yet they are distinct, the distinction lying in the part taken in the instability by viscosity.

If we imagine a fluid free from viscosity and absolutely free to glide over solid surfaces, then comparing such a fluid with a viscous fluid in exactly the same motion—

(1.) The frictionless fluid might be instable and the viscous fluid stable. Under these circumstances the cause of eddies is the instability as a perfect fluid, the effect of viscosity being in the direction of stability.

(2.) The frictionless fluid might be stable and the viscous fluid unstable, under which circumstances the cause of instability would be the viscosity.

It was clear to me that the conclusions I had drawn from the equations of motion immediately related only to the first cause; nor could I then perceive any possible way in which instability could result from viscosity. All the same I felt a certain amount of uncertainty in assuming the first cause of instability to be general. This uncertainty was the result of various considerations, but particularly from my having observed that eddies apparently come on in very different ways, according to a very definite circumstance of motion, which may be illustrated.

When in a channel the water is all moving in the same direction, the velocity being greatest in the middle and diminishing to zero at the sides, as indicated by the curve in fig. 1, eddies showed themselves reluctantly and irregularly; whereas when the

Fig. 1.

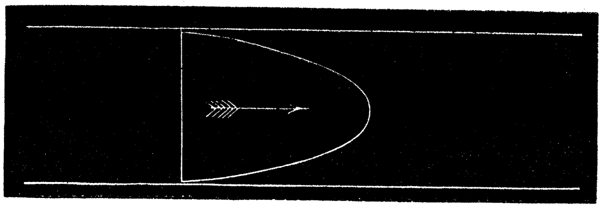
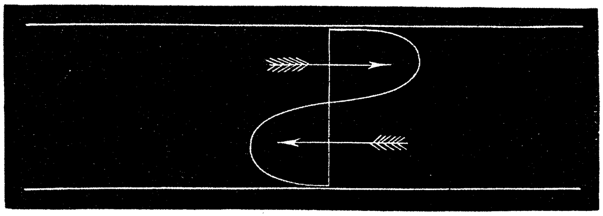


Fig. 2.



water on one side of the channel was moving in the opposite direction to that on the other, as shown by the curve in fig. 2, eddies appeared in the middle regularly and readily.

8. *Methods of investigation.*—There appeared to be two ways of proceeding—the one theoretical, the other practical.

The theoretical method involved the integration of the equations for unsteady motion in a way that had not been accomplished and which, considering the general intractability of the equations, was not promising.

The practical method was to test the relation between U, $\frac{\mu}{\rho}$, and c; this, owing to the simple and definite form of the law, seemed to offer, at all events in the first place, a far more promising field of research.

The law of motion in a straight smooth tube offered the simplest possible circumstances and the most crucial test.

The existing experimental knowledge of the resistance of water in tubes, although very extensive, was in one important respect incomplete. The previous experiments might be divided into two classes: (1) those made under circumstances in which the law of resistance was as the square of the velocity, and (2) those made under circumstances in which the resistance varied as the velocity. There had not apparently been any attempt made to determine the exact circumstances under which the change of law took place.

Again, although it had been definitely pointed out that eddies would explain resistance as the square of the velocity, it did not appear that any definite experimental evidence of the existence of eddies in parallel tubes had been obtained, and much less was there any evidence as to whether the birth of eddies was simultaneous with the change in the law of resistance.

These open points may be best expressed in the form of queries to which the answers anticipated were in the affirmative.

(1.) What was the exact relation between the diameters of the pipes and the velocities of the water at which the law of resistance changed?

Was it at a certain value of

$$c\mathrm{U}?$$

(2.) Did this change depend on the temperature, *i.e.*, the viscosity of water? Was it at a certain value of

$$\rho\frac{\mathrm{U}}{\mu}?$$

(3.) Were there eddies in parallel tubes?

(4.) Did steady motion hold up to a critical value and then eddies come in?

(5.) Did the eddies come in at a certain value of

$$\frac{\rho c\mathrm{U}}{\mu}?$$

(6.) Did the eddies first make their appearance as small and then increase gradually with the velocity, or did they come in suddenly?

The bearing of the last query may not be obvious; but, as will appear in the sequel, its importance was such that, in spite of satisfactory answers to all the other queries, a negative answer to this, in respect of one particular class of motions, led me to the reconsideration of the supposed cause of instability.

The queries, as they are put, suggest two methods of experimenting :—

(1.) Measuring the resistances and velocities of different diameters, and with different temperatures of water.

(2.) Visual observation as to the appearance of eddies during the flow of water along tubes or open channels.

Both these methods have been adopted, but, as the questions relating to eddies had been the least studied, the second method was the first adopted.

9. *Experiments by visual observation.*—The most important of these experiments related to water moving in one direction along glass tubes. Besides this, however, experiments on fluids flowing in opposite directions in the same tube were made, also a third class of experiments, which related to motion in a flat channel of indefinite breadth.

These last-mentioned experiments resulted from an incidental observation during some experiments made in 1876 as to the effect of oil to prevent wind waves. As the result of this observation had no small influence in directing the course of this investigation, it may be well to describe it first.

10. *Eddies caused by the wind beneath the oiled surface of water.*—A few drops of oil on the windward side of a pond during a stiff breeze, having spread over the pond and completely calmed the surface as regards waves, the sheet of oil, if it may be so called, was observed to drift before the wind, and it was then particularly noticed that while close to, and for a considerable distance from the windward edge, the surface presented the appearance of *plate glass;* further from the edge the surface presented that irregular wavering appearance which has already been likened to that of sheet glass, which appearance was at the time noted as showing the existence of eddies beneath the surface.

Subsequent observation confirmed this first view. At a sufficient distance from the windward edge of an oil-calmed surface there are always eddies beneath the surface even when the wind is light. But the distance from the edge increases rapidly as the force of the wind diminishes, so that at a limited distance (10 or 20 feet) the eddies will come and go with the wind.

Without oil I was unable to perceive any indication of eddies. At first I thought that the waves might prevent their appearance even if they were there, but by careful observation I convinced myself that they were not there. It is not necessary to discuss these results here, although, as will appear, they have a very important bearing on the cause of instability.

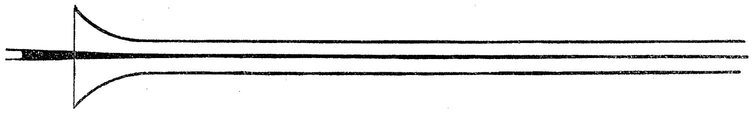
11. *Experiments by means of colour bands in glass tubes.*—These were undertaken early in 1880 ; the final experiments were made on three tubes, Nos. 1, 2, and 3. The diameters of these were nearly 1 inch, $\frac{1}{2}$ inch, and $\frac{1}{4}$ inch. They were all about 4 feet 6 inches long, and fitted with trumpet mouthpieces, so that water might enter without disturbance.

The water was drawn through the tubes out of a large glass tank, in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.

The general results were as follows :—

(1.) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, fig. 3.

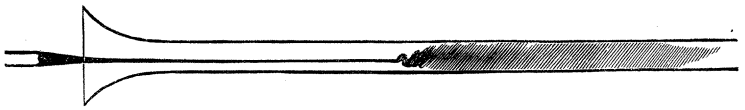
Fig. 3.



(2.) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

(3.) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in fig. 4.

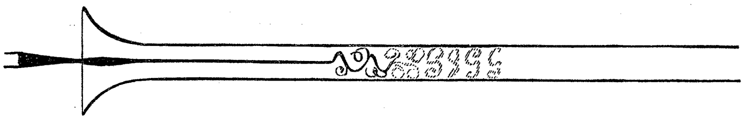
Fig. 4.



Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in fig. 5.

Fig. 5.



The experiments thus seemed to settle questions 3 and 4 in the affirmative, the existence of eddies and a critical velocity.

They also settled in the negative question 6, as to the eddies coming in gradually after the critical velocity was reached.

In order to obtain an answer to question 5, as to the law of the critical velocity, the diameters of the tubes were carefully measured, also the temperature of the water, and the rate of discharge.

(4.) It was then found that, with water at a constant temperature, and the tank as still as could by any means be brought about, the critical velocities at which the eddies showed themselves were almost exactly in the inverse ratio of the diameters of the tubes.

(5.) That in all the tubes the critical velocity diminished as the temperature increased, the range being from 5° C. to 22° C.; and the law of this diminution, so far as could be determined, was in accordance with POISEUILLE'S experiments. Taking T to express degrees centigrade, then by POISEUILLE'S experiments,

$$\frac{\mu}{\rho} \propto P = (1 + 0{\cdot}0336\,T + 0{\cdot}00221\,T^2)^{-1}$$

taking a metre as the unit, U_s the critical velocity, and D the diameter of the tube, the law of the critical point is completely expressed by the formula

$$U_s = \frac{1}{B_s}\,\frac{P}{D}$$

where

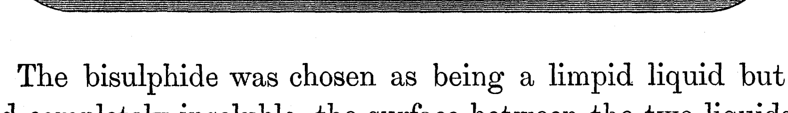
$$B_s = 43{\cdot}79$$
$$\log B_s = 1{\cdot}64139$$

This is a complete answer to question 5.

During the experiments many things were noticed which cannot be mentioned here, but two circumstances should be mentioned as emphasizing the negative answer to question 6. In the first place, the critical velocity was much higher than had been expected in pipes of such magnitude, resistance varying as the square of the velocity had been found at very much smaller velocities than those at which the eddies appeared when the water in the tank was steady; and in the second place, it was observed that the critical velocity was very sensitive to disturbance in the water before entering the tubes; and it was only by the greatest care as to the uniformity of the temperature of the tank and the stillness of the water that consistent results were obtained. This showed that the steady motion was unstable for large disturbances long before the critical velocity was reached, a fact which agreed with the full-blown manner in which the eddies appeared.

12. *Experiments with two streams in opposite directions in the same tube.*—A glass tube, 5 feet long and 1·2 inch in diameter, having its ends slightly bent up, as shown in fig. 6, was half filled with bisulphide of carbon, and then filled up with water and both

Fig. 6.



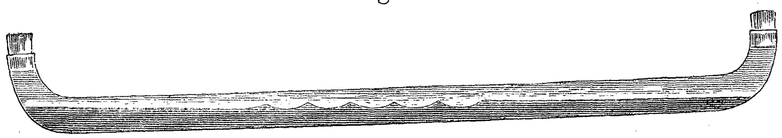
ends corked. The bisulphide was chosen as being a limpid liquid but little heavier than water and completely insoluble, the surface between the two liquids being clearly distinguishable. When the tube was placed in a horizontal direction, the weight of the bisulphide caused it to spread along the lower half of the tube, and the surface of separation of the two liquids extended along the axis of the tube. On one end of the tube being slightly raised the water would flow to the upper end and the bisulphide fall to the lower, causing opposite currents along the upper and lower halves of the tube, while in the middle of the tube the level of the surface of separation remained unaltered.

The particular purpose of this investigation was to ascertain whether there was a critical velocity at which waves or sinuosities would show themselves in the surface of separation.

It proved a very pretty experiment and completely answered its purpose.

When one end was raised quickly by a definite amount, the opposite velocities of the two liquids, which were greatest in the middle of the tube, attained a certain maximum value, depending on the inclination given to the tube. When this was small no signs of eddies or sinuosities showed themselves; but, at a certain definite inclination, waves (nearly stationary) showed themselves, presenting all the appearance of wind waves. These waves first made their appearance as very small waves of equal lengths, the length being comparable to the diameter of the tube.

Fig. 7.



When by increasing the rise the velocities of flow were increased, the waves kept the same length but became higher, and when the rise was sufficient the waves would curl and break, the one fluid winding itself into the other in regular eddies.

Whatever might be the cause, a skin formed slowly between the bisulphide and the water, and this skin produced similiar effects to that of oil on water; the results mentioned are those which were obtained before the skin showed itself. When the skin first came on regular waves ceased to form, and in their place the surface was disturbed, as if by irregular eddies, above and below, just as in the case of the oiled surface of water.

The experiment was not adapted to afford a definite measure of the velocities at which the various phenomena occurred; but it was obvious that the critical velocity at which the waves first appeared was many times smaller than the critical velocity in a tube of the same size when the motion was in one direction only. It was also clear that the critical velocity was nearly, if not quite, independent of any existing disturbance in the liquids; so that this experiment shows—

(1.) That there is a critical velocity in the case of opposite flow at which direct motion becomes unstable.

(2.) That the instability came on gradually and did not depend on the magnitude of the disturbances, or in other words, that for this class of motion question 6 must be answered in the affirmative.

It thus appeared that there was some difference in the cause of instability in the two motions.

13. *Further study of the equations of motion.*—Having now definite data to guide me, I was anxious to obtain a fuller explanation of these results from the equations of motion. I still saw only one way open to account for the instability, namely, by assuming the instability of a frictionless fluid to be general.

Having found a method of integrating the equations for frictionless fluid as far as to show whether any particular form of steady motion is stable for a small disturbance, I applied this method to the case of parallel flow in a *frictionless* fluid. The result, which I obtained at once, was that flow in one direction was stable, flow in opposite directions unstable. This was not what I was looking for, and I spent much time in trying to find a way out of it, but whatever objections my method of integration may be open to, I could make nothing less of it.

It was not until the end of 1882 that I abandoned further attempts with a frictionless fluid, and attempted by the same method the integration of a viscous fluid. This change was in consequence of a discovery that in previously considering the effect of viscoscity I had omitted to take fully into account the boundary conditions which resulted from the friction between the fluid and the solid boundary.

On taking these boundary conditions into account, it appeared that although the tendency of viscoscity through the fluid is to render direct or steady motion stable, yet owing to the boundary condition resulting from the friction at the solid surface, the motion of the fluid, irrespective of viscoscity, would be unstable. Of course this cannot be rendered intelligible without going into the mathematics. But what I want to point out is that this instability, as shown by the integration of the equations of motion, depends on exactly the same relation

$$U \propto \frac{\mu \rho}{}$$

as that previously found.

This explained all the practical anomalies and particularly the absence of eddies below a pure surface of water exposed to the wind. For in this case the surface being free, the boundary condition was absent, whereas the film of oil, by its tangential stiffness, introduced this condition; this circumstance alone seemed a sufficient verification of the theoretical conclusion.

But there was also the sudden way in which eddies came into existence in the experiments with the colour band, and the effect of disturbances to lower the critical velocity. These were also explained, for as long as the motion was steady, the instability depended upon the boundary action alone, but once eddies were introduced, the stability would be broken down.

It thus appeared that the meaning of the experimental results had been ascertained, and the relation between the four leading features and the circumstances on which they depend traced for the case of water in parallel flow.

But as it appeared that the critical velocity in the case of motion in one direction did not depend on the cause of instability with a view to which it was investigated, it followed that there must be another critical velocity, which would be the velocity at which previously existing eddies would die out, and the motion become steady as the water proceeded along the tube. This conclusion has been verified.

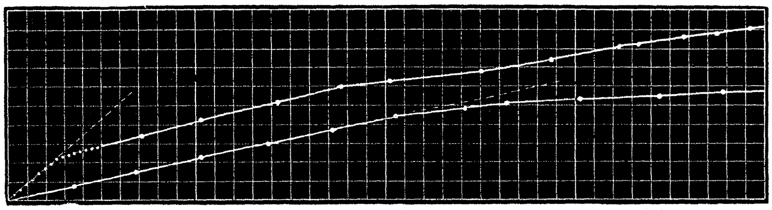
14. *Results of experiments on the law of resistance in tubes.*—The existence of the critical velocity described in the previous article could only be tested by allowing water in a high state of disturbance to enter a tube, and after flowing a sufficient distance for the eddies to die out, if they were going to die out, to test the motion.

As it seemed impossible to apply the method of colour bands, the test applied was that of the law of resistance as indicated in questions (1) and (2) in § 8. The result was very happy.

Two straight lead pipes No. 4 and No. 5, each 16 feet long and having diameters of a quarter and a half inch respectively were used. The water was allowed to flow through rather more than 10 feet before coming to the first gauge hole, the second gauge hole being 5 feet further along the pipe.

The results were very definite, and are partly shown in fig. 8, and more fully in diagram 1, Plate 74.

Fig. 8.



(1.) At the lower velocities the pressure was proportional to the velocity, and the velocities at which a deviation from the law first occurred were in exact inverse ratio of the diameters of the pipes.

(2.) Up to these critical velocities the discharge from the pipes agreed exactly with those given by POISEUILLE's formula for capillary tubes.

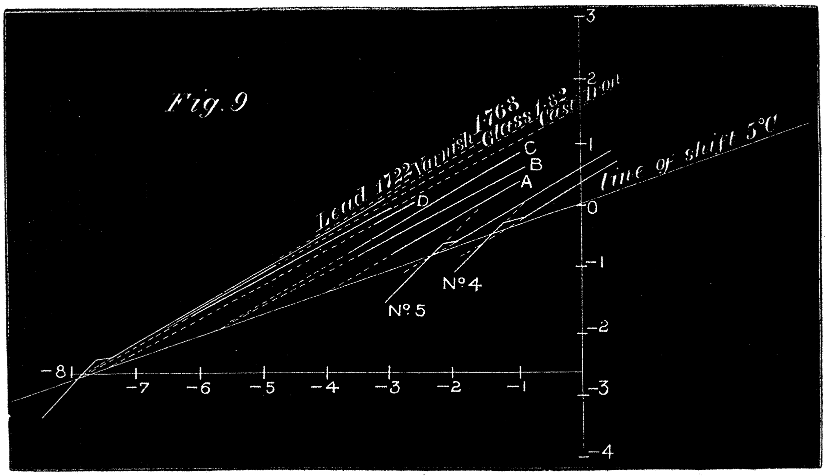
(3.) For some little distance after passing the critical velocity, no very simple relations appeared to hold between the pressures and velocities. But by the time the velocity reached 1·2 (critical velocity) the relation became again simple. The pressure did not vary as the square of the velocity, but as 1·722 power of the velocity, this law held in both tubes and through velocities ranging from 1 to 20, where it showed no signs of breaking down.

(4.) The most striking result was that not only at the critical velocity, but throughout the entire motion, the laws of resistance exactly corresponded for velocities in the ratio of

$$\frac{\mu}{\rho c}$$

This last result was brought out in the most striking manner on reducing the results by the graphic method of logarithmic homologues as described in my paper on Thermal Transpiration. Calling the resistance per unit of length as measured in the weight of cubic units of water i, and the velocity v, log i is taken for abscissa, and log v for ordinate, and the curve plotted.

In this way the experimental results for each tube are represented as a curve; these curves, which are shown as far as the small scale will admit in fig. 9, present exactly the same shape, and only differ in position.



Pipe.	Diameter.
	m.
No. 4, Lead	0·00615
„ 5, „	0·0127
A, Glass	0·0496
B, Cast iron	0·188
D, „	0·5
C, Varnish	0·196

Either of the curves may be brought into exact coincidence with the other by a rectangular shift, and the horizontal shifts are given by the difference of the logarithm of

$$\frac{D^3}{\mu^2}$$

for the two tubes, the vertical shifts being the difference of the logarithms of

$$\frac{D}{\mu}$$

The temperatures at which the experiment had been made were nearly the same, but not quite, so that the effect of the variations of μ showed themselves.

15. *Comparison with* DARCY'S *experiments.*—The definiteness of these results, their agreement with POISEUILLE'S law, and the new form which they more than indicated for the law of resistance above the critical velocities, led me to compare them with

the well known experiments of DARCY on pipes ranging from 0·014 to 0·5 metre in diameter.

Taking no notice of the empirical laws by which DARCY had endeavoured to represent his results, I had the logarithmic homologues drawn from his published experiments. If my law was general then these logarithmic curves, together with mine, should all shift into coincidence, if each were shifted horizontally through

$$\frac{D^3}{P^2}$$

and vertically through

$$\frac{D}{P}$$

In calculating these shifts there were some doubtful points. DARCY'S pipes were not uniform between the gauge points, the sections varying as much as 20 per cent., and the temperature was only casually given. These matters rendered a close agreement unlikely. It was rather a question of seeing if there was any systematic disagreement. When the curves came to be shifted the agreement was remarkable. In only one respect was there any systematic disagreement, and this only raised another point; it was only in the slopes of the higher portions of the curves. In both my tubes the slopes were as 1·722 to 1; in DARCY'S they varied according to the nature of the material, from the lead pipes, which were the same as mine, to 1·92 to 1 with the cast iron.

This seems to show that the nature of the surface of the pipe has an effect on the law of resistance above the critical velocity.

16. *The critical velocities.*—All the experiments agreed in giving

$$v_c = \frac{1}{278}\frac{P}{D}$$

as the critical velocity, to which corresponds as the critical slope of pressure

$$i_c = \frac{1}{47700000}\frac{P^2}{D^3}$$

the units being metres and degrees centigrade. It will be observed that this value is much less than the critical velocity at which steady motion broke down; the ratio being 43·7 to 278.

17. *The general law of resistance.*—The logarithmic homologues all consist of two straight branches, the lower branch inclined at 45 degrees and the upper one at n horizontal to 1 vertical. Except for the small distance beyond the critical velocity these branches constitute the curves. These two branches meet in a point on the curve at a definite distance below the critical pressure, so that, ignoring the small portion of the curve above the point before it again coincides with the upper branch, the logarithmic homologue gives for the law of resistance for all pipes and all velocities

$$A\frac{D^3}{\theta^2}i=(B\frac{D}{\theta}v)^n$$

where n has the value unity as long as either number is below unity, and then takes the value of the slope n to 1 for the particular surface of the pipe.

If the units are metres and degrees centigrade

$$A=67,700,000$$
$$B=396$$
$$P=(1+0\cdot0336\ T+0\cdot000221\ T^2)^{-1}$$

This equation then, excluding the region immediately about the critical velocity, gives the law of resistance in POISEUILLE's tubes, those of the present investigation and DARCY's, the range of diameters being

from 0·000013 (POISEUILLE, 1845)
to 0·5 (DARCY, 1857)

and the range of velocities

from 0·0026 } metres per sec., 1883.
to 7·

This algebraical formula shows that the experiments entirely accord with the theoretical conclusions.

The empirical constants are A, B, P, and n; the first three relate solely to the dimensional properties of the fluid summed up in the viscosity, and it seems probable that the last relates to the properties of the surface of the pipe.

Much of the success of the experiments is due to the care and skill of Mr. FOSTER, of Owens College, who has constructed the apparatus and assisted me in making the experiments.

SECTION II.

Experiments in glass tubes by means of colour bands.

18. In commencing these experiments it was impossible to form any very definite idea of the velocity at which eddies might make their appearance with a particular tube. The experiments of POISEUILLE showed that the law of resistance varying as the velocity broke down in a pipe of say 0·6 millim. diameter; and the experiments of DARCY showed this law did not hold in a half-inch pipe with a velocity of 6 inches per second.

These considerations, together with the comparative ease with which experiments on a small scale can be made, led me to commence with the smallest tube in which I could hope to perceive what was going on with the naked eye, expecting confidently that eddies would make their appearance at an easily attained velocity.

19. *The first apparatus.*—This consisted of a tube about $\frac{1}{4}$ inch or 6 millims. in diameter. This was bent into the siphon form having one straight limb about 2 feet long and the other about 5 feet (Plate 72, fig. 10).

The end of the shorter limb was expanded to a bell mouth, while the longer end was provided with an indiarubber extension on which was a screw clip.

The bell-mouthed limb was held vertically in the middle of a beaker with the mouth several inches from the bottom as shown in figs. 10 and 10′.

A colour tube about 6 millims. in diameter also of siphon form was placed as shown in the figure, with the open end of the shorter limb just under the bell mouth, the longer limb communicating through a controlling clip with a reservoir of highly coloured water placed at a sufficient height. A supply-pipe was led into the beaker for the purpose of filling it; but not with the idea of maintaining it full, as it seemed probable that the inflowing water would create too much disturbance, experience having shown how important perfect internal rest is to successful experiments with coloured water.

20. *The first experiment.*—The vessels and the siphons having been filled and allowed to stand for some hours so as to allow all internal notion to cease, the colour clip was opened so as to allow the colour to emerge slowly below the bell (Plate 73, fig. 11).

Then the clip on the running pipe was opened very gradually. The water was drawn in at the bell mouth, and the coloured water entered, at first taking the form of a candle flame (Plate 73, fig. 12), which continually elongated until it became a very fine streak, contracting immediately on leaving the colour-tube and extending all along the tube from the bell mouth to the outlet (fig. 10). On further opening the regulating clip so as to increase the velocity of flow, the supply of colour remaining unaltered, the only effect was to diminish the thickness of the colour band. This was again increased by increasing the supply of colour, and so on until the velocity was the greatest that circumstances would allow—until the clip was fully open. Still the colour band was perfectly clear and definite throughout the tube. It was apparent that if there were to be eddies it must be at a higher velocity. To obtain this about 2 feet more were added to the longer leg of the siphon which brought it down to the floor.

On trying the experiment with this addition the colour band was still clear and undisturbed.

So that for want of power to obtain greater velocity this experiment failed to show eddies.

When the supply pipe which filled the beaker was kept running during the experiment, it kept the water in the beaker in a certain state of disturbance. The effect of this disturbance was to disturb the colour band in the tube, but it was extremely difficult to say whether this was due to the wavering of the colour band or to genuine eddies.

21. *The final apparatus.*—This was on a much larger scale than the first. A straight tube, nearly 5 feet long and about an inch in diameter, was selected from a large number as being the most nearly uniform, the variation of the diameter being less than 1-32nd of an inch.

The ends of this tube were ground off plane, and on the end which appeared slightly the larger was fitted a trumpet mouth of varnished wood, great care being taken to make the surface of the wood continuous with that of the glass (Plate 73, fig. 13).

The other end of the glass pipe was connected by means of an indiarubber washer with an iron pipe nearly 2 inches in diameter.

The iron pipe passed horizontally through the end of a tank, 6 feet long, 18 inches broad and 18 inches deep, and then bent through a quadrant so that it became vertical, and reached 7 feet below the glass tube. It then terminated in a large cock, having, when open, a clear way of nearly a square inch.

This cock was controlled by a long lever (see Plate 73) reaching up to the level of the tank. The tank was raised upon tressels about 7 feet above the floor, and on each side of it, at 4 feet from the ground, was a platform for the observers. The glass tube thus extended in an horizontal direction along the middle of the tank, and the trumpet mouth was something less than a foot from the end. Through this end, just opposite the trumpet, was a straight colour tube three-quarters of an inch in diameter, and this tube was connected, by means of an indiarubber tube with a clip upon it, with a reservoir of colour, which for good reasons subsequently took the form of a common water bottle.

With a view to determining the velocity of flow, an instrument was fitted for showing the changes of level of the water in the tank to the 100th of an inch (Plate 72, fig. 14). Thermometers were hung at various levels in the tank.

22. *The final experiments.*—The first experiment with this apparatus was made on 22nd February, 1880.

By means of a hose the tank was filled from the water main, and having been allowed to stand for several hours, from 10 A.M. to 2 P.M., it was then found that the water had a temperature of 46° F. at the bottom of the tank, and 47° F. at the top. The experiment was then commenced in the same manner as in the first trials. The colour was allowed to flow very slowly, and the cock slightly opened. The colour band established itself much as before, and remained beautifully steady as the velocity was increased until, all at once, on a slight further opening of the valve, at a point about two feet from the iron pipe, the colour band appeared to expand and mix with the water so as to fill the remainder of the pipe with a coloured cloud, of what appeared at first sight to be of a uniform tint (fig. 4, p. 942).

Closer inspection, however, showed the nature of this cloud. By moving the eye so as to follow the motion of the water, the expansion of the colour band resolved itself into a well-defined waving motion of the band, at first without other disturbance, but after two or three waves came a succession of well-defined and distinct eddies. These were sufficiently recognisable by following them with the eye, but more distinctly seen by a flash from a spark, when they appeared as in fig. 5, p. 942.

The first time these were seen the velocity of the water was such that the tank fell 1 inch in 1 minute, which gave a velocity of $0^{m}\cdot627$, or 2 feet per second. On slightly closing the valve the eddies disappeared, and the straight colour band established itself.

Having thus proved the existence of eddies, and that they came into existence at a certain definite velocity, attention was directed to the relations between this critical velocity, the size of the tube, and the viscosity.

Two more tubes (2 and 3) were prepared similar in length and mounting to the first, but having diameters of about one-half and one-quarter inch respectively.

In the meantime an attempt was made to ascertain the effect of viscosity by using water at different temperatures. The temperature of the water from the main was about 45°, the temperature of the room about 54°; to obtain a still higher temperature, the tank was heated to 70° by a jet of steam. Then taking, as nearly as we could tell, similar disturbances, the experiments which are numbered 1 and 2 in Table I. were made.

To compare these for the viscosity, POISEUILLE's experiments were available, but to prevent any accidental peculiarity of the water being overlooked, experiments after the same manner as POISEUILLE's were made with the water in the tank. The results of these however agreed so exactly with those of POISEUILLE that the comparative effect of viscosity was taken from POISEUILLE's formula

$$P^{-1}=1+0\cdot03368\,T+0\cdot000221\,T^2$$

where $P\propto\mu$ with the temperature and T is temperature centigrade.

The relative values of P at 47° and 70° Fah. are as

$$1\cdot3936 \text{ to } 1$$

while the relative critical velocities at these temperatures were as

$$1\cdot45 \text{ to } 1$$

which agreement is very close considering the nature of the experiments.

But whatever might have been the cause of the previous anomalies, these were greatly augmented in the heated tank. After being heated the tank had been allowed to stand for an hour or two, in order to become steady. On opening the valve it was thought that the eddies presented a different appearance from those in the colder water, and the thought at once suggested itself that this was due to some source of initial disturbance. Several sources of such disturbance suggested themselves—the temperature of the tank was 11° C. above that of the room, and the cooling arising from the top and sides of the tank must cause circulation in the tank. A few streaks of colour added to the water soon showed that such a circulation existed, although it was very slow. Another source of possible disturbance was the difference in the temperature at the top and bottom of the tank, this had been as much as 5°.

In order to get rid of these sources of disturbance it was necessary to have the tank at the same temperature as the room, about 54° or 55°. Then it was found by several trials that the eddies came on at a fall of about 1 inch in 64 seconds, which, taking the viscosity into account, was higher than in the previous case, and this was taken to indicate that there was less disturbance in the water.

As it was difficult to alter the temperatures of the building so as to obtain experiments under like conditions at a higher temperature, and it appeared that the same object would be accomplished by cooling the water to its maximum density, 40°, this plan was adopted and answered well, ice being used to cool the water.

Experiments were then made with three tubes 1, 2, 3, at temperatures of about 51° and 40°. All are given in Table I.

TABLE I.

Experiments with Colour Bands—Critical Velocities at which Steady Motion breaks down.

Pipe No. 1, glass.—Diameter 0·0268 metre; log diameter $\bar{2}$·42828.
,, No. 2, ,, ,, 0·01527 ,, ,, $\bar{2}$·18400.
,, No. 3, ,, ,, 0·007886 ,, ,, $\bar{3}$·89783.
Discharge, cub. metre=·021237; log =$\bar{2}$·32709.

Date, 1880.	Reference Number.	Pipe.	Temperature, centigrade.	Time of discharge.	Velocity, metres.	log time.	−log P.	log V.	log B_s.
1 March	1	No. 1.	8·3	60	0·6270	1·77815	0·11242	$\bar{1}$·79729	1·66200
3 ,,	2	,,	21	87	0·4325	1·93959	0·25654	$\bar{1}$·63593	1·67930
25 ,,	3	,,	15	70	0·5374	1·84500	0·19198	$\bar{1}$·73035	1·64936
21 April	4	,,	12	60	0·6270	1·77815	0·15712	$\bar{1}$·79729	1·61730
,,	5	,,	13	64	0·5878	1·80618	0·16882	$\bar{1}$·76926	1·64464
,,	6	,,	13	67	0·5614	1·82617	0·16882	$\bar{1}$·74927	1·65363
,,	7	,,	13	64	0·5878	1·80618	0·16882	$\bar{1}$·76926	1·64464
,,	8	,,	5	54	0·6967	1·73239	0·06963	$\bar{1}$·84305	1·65898
,,	9	,,	5	52	0·7235	1·71600	0·06963	$\bar{1}$·85940	1·64269
22 ,,	10	,,	10	62	0·6068	1·79239	0·13319	$\bar{1}$·78305	1·65546
,,	11	,,	11	64	0·5870	1·80613	0·14525	$\bar{1}$·76931	1·65716
25 March	12	No. 2.	22	155	0·7476	2·19033	0·26710	$\bar{1}$·87367	1·67523
23 April	13	,,	11	110	1·052	2·04139	0·14525	0·02261	1·64814
,,	14	,,	11	108	1·072	2·03342	0·14525	0·03058	1·64017
,,	15	,,	4	83	1·396	1·91907	0·05621	0·14493	1·61486
,,	16	,,	4	83	1·396	1·91907	0·05621	0·14493	1·61486
,,	17	,,	4	83	1·396	1·91907	0·05621	0·14493	1·61486
,,	18	,,	6	86	1·348	1·93449	0·08278	0·12951	1·59371
,,	19	,,	6	85	1·362	1·92941	0·08278	0·13459	1·59863
24 ,,	20	No. 3.	11	220	1·967	2·34242	0·14525	0·29392	1·66300
,,	21	,,	10·5	224	1·932	2·35024	0·13920	0·28610	1·67687
,,	22	,,	11	218	1·982	2·33845	0·14525	0·29789	1·65903
,,	23	,,	11	116	2·004	2·33445	0·14525	0·30189	1·65503
25 ,,	24	,,	4	164	2·637	2·21484	0·05621	0·42150	1·62446
,,	25	,,	4	172	2·517	2·23552	0·05621	0·40082	1·64514
,,	26	,,	6	176	2·460	2·24551	0·08278	0·39083	1·62856
,,	27	,,	6	176	2·460	2·24551	0·08278	0·39083	1·62856
,,	28	,,	6	174	2·488	2·24054	0·08278	0·39580	1·62359
,,	29	,,	6	177	2·446	2·24791	0·08278	0·38837	1·63102

This gives the mean value for log B_s 1·64139; and B_s=43·79.

In reducing the results the unit taken has been the metre and the temperature is given in degrees centigrade.

The diameters of the three tubes were found by filling them with water.

The time measured was the time in which the tank fell 1 inch, which in cubic metres is given by

$$Q = \cdot 021237$$

In the table the logarithms of P, v, and B_s are given, as well as the natural numbers for the sake of reference.

The velocities v have been obtained by the formula

$$v = \frac{4\,Q}{\pi\,D^2}$$

B_s being obtained from the formula

$$B_s = \frac{P}{vD}$$

The final value of B_s is obtained from the mean value of the logarithm of B_s.

23. *The results.*—The values of log B_s show a considerable amount of regularity, and prove, I think conclusively, not only the existence of a critical velocity at which eddies come in, but that it is proportional to the viscosity and inversely proportional to the diameter of the tube.

The fact, however, that this relation has only been obtained by the utmost care to reduce the internal disturbances in the water to a minimum must not be lost sight of.

The fact that the steady motion breaks down suddenly shows that the fluid is in a state of instability for disturbances of the magnitude which cause it to break down. But the fact that in some conditions it will break down for a large disturbance, while it is stable for a smaller disturbance shows that there is a certain residual stability so long as the disturbances do not exceed a given amount.

The only idea that I had formed before commencing the experiments was that at some critical velocity the motion must become unstable, so that any disturbance from perfectly steady motion would result in eddies.

I had not been able to form any idea as to any particular form of disturbance being necessary. But experience having shown the impossibility of obtaining absolutely steady motion, I had not doubted but that appearance of eddies would be almost simultaneous with the condition of instability. I had not, therefore, considered the disturbances except to try and diminish them as much as possible. I had expected to see the eddies make their appearance as the velocity increased, at first in a slow or feeble manner, indicating that the water was but slightly unstable. And it was a matter of surprise to me to see the sudden force with which the eddies sprang into existence, showing a highly unstable condition to have existed at the time the steady motion broke down.

This at once suggested the idea that the condition might be one of instability for disturbance of a certain magnitude and stable for smaller disturbances.

In order to test this, an open coil of wire was placed in the tube so as to create a definite disturbance as in Plate 72, fig. 15.

Eddies now showed themselves at a velocity of less than half the previous critical velocity, and these eddies broke up the colour band, but it was difficult to say whether the motion was really unstable or whether the eddies were the result of the initial disturbance, for the colour band having once broken up and become mixed with the water, it was impossible to say whether the motion did not tend to become steady again later on in the tube.

Subsequent observation however tended to show that the critical value of the velocity depended to some extent on the initial steadiness of the water. One phenomenon in particular was very marked.

Where there was any considerable disturbance in the water of the tank and the cock was opened very gradually, the state of disturbance would first show itself by the wavering about of the colour band in the tube; sometimes it would be driven against the glass and would spread out, and all without a symptom of eddies. Then, as the velocity increased but was still comparatively small, eddies, and often very regular eddies, would show themselves along the latter part of the tube. On further opening the cock these eddies would disappear and the colour band would become fixed and steady right through the tube, which condition it would maintain until the velocity reached its normal critical value, and then the eddies would appear suddenly as before.

Another phenomenon very marked in the smaller tubes, was the intermittent character of the disturbance. The disturbance would suddenly come on through a certain length of the tube and pass away and then come on again, giving the appearance of flashes, and these flashes would often commence successively at one point in the pipe. The appearance when the flashes succeeded each other rapidly was as shown in Plate 72, fig. 16.

This condition of flashing was quite as marked when the water in the tank was very steady as when somewhat disturbed.

Under no circumstances would the disturbance occur nearer to the trumpet than about 30 diameters in any of the pipes, and the flashes generally, but not always, commenced at about this distance.

In the smaller tubes generally, and with the larger tube in the case of the ice-cold water at 40°, the first evidence of instability was an occasional flash beginning at the usual place and passing out as a disturbed patch two or three inches long. As the velocity was further increased these flashes became more frequent until the disturbance became general.

I did not see a way to any very crucial test as to whether the steady motion became unstable for a large disturbance before it did so for a small one; but the general impression left on my mind was that it did in some way—as though disturbances in the tank, or arising from irregularities in the tube, were necessary to the existence of a state of instability.

But whatever these peculiarities may mean as to the way in which eddies present themselves, the broad fact of there being a critical value for the velocity at which the steady motion becomes unstable, which critical value is proportional to

$$\frac{\mu}{\rho c}$$

where c is the diameter of the pipe and $\frac{\mu}{\rho}$ the viscosity by the density, is abundantly established. And cylindrical glass pipes for approximately steady water have for the critical value

$$v = \frac{P}{B_s D}$$

where in metres $B_s = 43{\cdot}79$ about.

Section III.

Experiments to determine the critical velocity by means of resistance in the pipes.

24. Although at first sight such experiments may appear to be simple enough, yet when one began to consider actual ways and means, so many uncertainties and difficulties presented themselves that the necessary courage for undertaking them was only acquired after two years' further study of the hydrodynamical aspect of the subject by the light thrown upon it by the previous experiment with the colour bands. This has been already explained in Art. 13. Those experiments had shown definitely that there was a critical value of the velocity at which eddies began if the water were approximately steady when drawn into the tube, but they had also shown definitely that at such critical velocity the water in the tube was in a highly unstable condition, any considerable disturbance in the water causing the break down to occur at velocities much below the highest that could be attained when the water was at its steadiest; suggesting that if there were a critical velocity at which, for any disturbance whatever, the water became stable, this velocity was much less than that at which it would become unstable for infinitely small disturbances; or, in other words, suggesting that there were two critical values for the velocity in the tube, the one at which steady motion changed into eddies, the other at which eddies changed into steady motion.

Although the law for the critical value of the velocity had been suggested by the equations of motion, it was, as already explained, only at the beginning of this year that I succeeded in dealing with these equations so as to obtain any theoretical explanation of the dual criteria; but having at last found this, it became clear to me that if in a tube of sufficient length the water were at first admitted in a high state of disturbance, then as the water proceeded along the tube the disturbance would settle down into a steady condition, which condition would be one of eddies or steady motion, according to whether the velocity was above or below what may be called the real critical value.

The necessity of initial disturbance precluded the method of colour bands, so that the only method left was to measure the resistance at the latter portion of the tube in conjunction with the discharge.

The necessary condition was somewhat difficult to obtain. The change in the law of resistance could only be ascertained by a series of experiments which had to be carried out under similar conditions as regards temperature, kind of water, and condition of the pipe; and in order that the experiments might be satisfactory, it seemed necessary that the range of velocities should extend far on each side of the critical velocity. In order to best ensure these conditions, it was resolved to draw the water direct from the Manchester main, using the pressure in the main for forcing the water through the pipes. The experiments were conducted in the workshop in Owens College, which offered considerable facilities owing to arrangements for supplying and measuring the water used in experimental turbines.

25. *The apparatus.*—This is shown in Plate 72, fig. 17.

As the critical value under consideration would be considerably below that found for the change for steady motion into eddies, a diameter of about half an inch (12 millims.) was chosen for the larger pipe, and one quarter of an inch for the smaller, such pipes being the smallest used in the previous experiments.

The pipes (4 and 5) were ordinary lead gas or water pipes. These, which owing to their construction are very uniform in diameter and when new present a bright metal surface inside, seemed well adapted for the purpose.

Pipes 4 (which was a quarter-inch pipe) and 5 (which was a half-inch) were 16 feet long, straightened by laying them in a trough formed by two inch boards at right angles. This trough was then fixed so that one side of the trough was vertical and the other horizontal, forming a horizontal ledge on which the pipes could rest at a distance of 7 feet from the floor; on the outflow ends of the pipes cocks were fitted to control the discharge, and at the inlet end the pipes were connected, by means of a T branch, with an indiarubber hose from the main; this connexion was purposely made in such a manner as to necessitate considerable disturbance in the water entering the pipes from the hose. The hose was connected, by means of a quarter-inch cock, with a four-inch branch from the main. With this arrangement the pressure on the inlet to the pipes was under control of the cock from the main, and at the same time the discharge from the pipes was under control from the cocks on their ends.

This double control was necessary owing to the varying pressure in the main, and after a few preliminary experiments a third and more delicate control, together with a pressure gauge, were added, so as to enable the observer to keep the pressure in the hose, *i.e.*, on the inlets to the pipes, constant during the experiments.

This arrangement was accomplished by two short branches between the hose and the control cock from the main, one of these being furnished with an indiarubber mouthpiece with a screw clip upon it, so that part of the water which passed the cock might be allowed to run to waste, the other branch being connected with the lower end of a vertical glass tube, about 6 millims. in diameter and 30 inches long, having a bulb about 2 inches diameter near its lower extremity, and being closed by a similar bulb at its top.

This arrangement served as a delicate pressure gauge. The water entering at the lower end forced the air from the lower bulb into the upper, causing a pressure of about 30 inches of mercury. Any further rise increased this pressure by forcing the air in the tubes into the upper bulb, and by the weight of water in the tube. During an experiment the screw clip was continually adjusted, so as to keep the level of the water in the glass tube between the bulbs constant.

26. *The resistance gauges.*—Only the last 5 feet of the tube was used for measuring the resistance, the first 10 or 11 feet being allowed for the acquirement of a regular condition of flow.

It was a matter of guessing that 10 feet would be sufficient for this, but since, compared with the diameter, this length was double as great for the smaller tube, it was expected that any insufficiency would show itself in a greater irregularity of the results obtained with the larger tube, and as no such irregularity was noticed it appears to have been sufficient.

At distances of 5 feet near the ends of the pipe, two holes of about 1 millim. were pierced into each of the pipes for the purpose of gauging the pressures at these points of the pipes. As owing to the rapid motion of the water in the pipes past these holes, any burr or roughness caused in the inside of the pipe in piercing these holes would be apt to cause a disturbance in the pressure, it was very important that this should be avoided. This at first seemed difficult, as owing to the distance—5 feet—of one of the holes from the end of pipes of such small diameter the removal of a burr, which would be certain to ensue on drilling the holes from the outside, was difficult. This was overcome by the simple expedient suggested by Mr. Foster of drilling holes completely through the pipes and then plugging the side on which the drill entered. Trials were made, and it was found that the burr thus caused was very slight.

Before drilling the holes short tubes had been soldered to the pipes, so that the holes communicated with these tubes; these tubes were then connected with the limbs of a siphon gauge by indiarubber pipes.

These gauges were about 30 inches long; two were used, the one containing mercury, the other bisulphide of carbon.

These gauges were constructed by bending a piece of glass tube into a U form, so that the two limbs were parallel and at about one inch apart.

Glass tubes are seldom quite uniform in diameter, and there was a difference in the size of the limbs of both gauges, the difference being considerable in the case of the bisulphide of carbon.

The tubes were fixed to stands with carefully graduated scales behind them, so that the height of the mercury or carbon in each limb could be read. It had been anticipated that readings taken in this way would be sufficient. But it turned out to be desirable to read variations of level of the smallness of $\frac{1}{1000}$th of an inch or $\frac{1}{40}$th of a millimetre.

A species of cathetometer was used. This had been constructed for my experiments on Thermal Transpiration, and would read the position of the division surface of two fluids to $\frac{1}{10000}$th inch (Phil. Trans. 1879, p. 789).

The water was carefully brought into direct connexion with the fluid in the gauge, the indiarubber connexions facilitating the removal of all air.

27. *Means adopted in measuring the discharge.*—For many reasons it was very desirable to measure the rate of discharge in as short a time as possible.

For this purpose a species of orifice or weir gauge was constructed, consisting of a vertical tin cylinder two feet deep, having a flat bottom, being open at the top, with a diaphragm consisting of many thicknesses of fine wire gauze about two inches from the bottom; a tube connected the bottom with a vertical glass tube, the height of water in which showed the pressure of water on the bottom of the gauze; behind this tube was a scale divided so that the divisions were as the square roots of the height. Through the thin tin bottom were drilled six holes, one an eighth of an inch diameter, one a quarter of an inch, and four of half an inch.

These holes were closed by corks so that any one or any combination could be used. The combinations used were:

Gauge No. 1. The $\frac{1}{8}$ inch hole alone.
No. 2. The $\frac{1}{4}$ inch hole alone.
No. 3. A $\frac{1}{2}$ inch hole alone.
No. 4. Two $\frac{1}{2}$ inch holes.
No. 5. Four $\frac{1}{2}$ inch holes.

According to experience, the velocity with which water flows from a still vessel through a round hole in a thin horizontal plate is very nearly proportional to the area of the hole and the square root of the pressure, so that with any particular hole the relative quantities of water discharged would be read off at the variable height gauge. The accuracy of the gauge, as well as the absolute values of the readings, was checked by comparing the readings on the gauge with the time taken to fill vessels of known capacity. In this way coefficients for each one of the combinations 1, 2, 3, 4, 5 were obtained as follows:—

TABLE II.

No. of Gauge.	Readings on Gauge.	Time.	Quantity.	Coefficient.	Logarithmic coefficient.
		Seconds.	c.c.		
Gauge No. 1	19·55	61	1160	·966	$\bar{1}$·985
ib.	—	59	1160		
No. 2	5·3	54	1160	4·055	·608
ib.	15·3 full	—	A	4·055	—
No. 3	15	360	A	16·220	1·210
No. 4	15	178	A	32·440	1·511
No. 5	15	90	A	64·880	1·812

From this table it will be seen that the absolute values of the coefficients were obtained from experiments on the gauges No. 1 and No. 2, the coefficients for the gauges 3, 4, and 5 being determined by comparison of the times taken to fill a vessel of unknown capacity, which stands in the Table as A. The relative value of these coefficients came out sensibly proportional to the squares of the diameters of the apertures.

For the smaller velocities it was found that the gauge No. 1 was too large, and in order not to delay the experiment in progress, two glass flasks were used : these are distinguished as flasks (1) and (2); their capacities, as subsequently determined with care, were 303 and 1160 c.c. The discharge as measured by the times taken to fill these flasks are reduced to c.c. per second by dividing the capacities of the flasks by the times.

28. *The method of carrying out the experiments.*—This was generally as follows:— My assistant, Mr. FOSTER, had charge of the supply of water from the main, keeping the water in the pressure gauge at a fixed level.

The tap at the end of the tube to be experimented upon being closed, the zero reading of the gauge was carefully marked, and the micrometer adjusted so that the spider line was on the division of water and fluid in the left hand limb of the gauge. The screw was then turned through one entire revolution, which lowered the spider line one-fiftieth of an inch ; the tap at the end of the pipe was then adjusted until the fluid in the gauge came down to the spider line ; having found that it was steady there, the discharge was measured.

This having been done, the spider line was lowered by another complete revolution of the screw, the tap again adjusted, and so on, for about 20 readings, which meant about half an inch difference in the gauge. Then the readings were taken for every five turns of the screw until the limit of the range, about 2 inches, was reached. After this, readings were taken by simple observation of the scale attached to the gauge. In taking these readings the best plan was to read the position of the mercury or carbon in both limbs of the gauge, but this was not always done, some of the readings entered in the notes referred to one or other limb of the gauge, care having been taken to indicate which.

In the Tables III., IV., and V. of results appended, the noted readings are given and the letters *r*, *l*, and *b* signify whether the reading was on the right or left limb, or the sum of the readings on both limbs.

The readings marked *l* and *r* are reduced by the correction for the difference in the size of the limbs as well as the coefficient for the particular fluid in the gauge.

Thus it was found with the mercury tube that when the left limb had moved through 39 divisions on the scale the right had moved through 41, so that to obtain the sum of these readings the readings on the left, or those marked *l*, had to be multiplied by 2·05 and those on the right by 1·95.

With the bisulphide of carbon gauge, 11 divisions on the left caused 9 on the right, so that the correction for the reading on the left was 1·8 and on the right 2·2.

29. *Comparison of the pressure gauges.*—The pressures as marked by the gauges were reduced to the same standard by comparing the gauges; thus ·25 of the left limb of the mercury corresponded with 24 inches on both limbs of the bisulphide. Therefore to reduce the readings of the bisulphide of carbon to the same scale as those of the mercury they were multiplied by

$$\frac{\cdot 25 \times 20\cdot 5}{24} = 0\cdot 0213$$

This brought the readings of pressure to the same standard, *i.e.*, $\frac{1}{1000}$th of an inch of mercury, but these were further reduced by the factor 0·00032 to bring them to metres of water.

As it was convenient for the sake of comparison to obtain the differences of pressure per unit length of the pipe, the pressures in metres of water have been divided by 1·524, the length in metres between the gauge holes, and these reductions are included in the tables of results in the column headed *i*.

From the discharges as measured by the various gauges having been reduced to cubic centimetres, the mean velocity of the water was found by dividing by the area of the section of the pipe.

30. *Sections and diameters of the pipes.*—The areas were obtained by carefully measuring the diameters by means of fitting brass plugs into the pipes and then measuring the plugs. In this way the diameters were found to be—

Diameter, No. 4 pipe, ·242 inch, 6·15 millims.
,, No. 5 pipe, ·498 inch, 12·7 millims.

These gave the areas of the sections—

Section, No. 4 pipe, 29·7 square millims.
,, No. 5 pipe, 125 square millims.

The discharge in cubic centimetres divided by the area of section in square millimetres gave the mean velocity in metres per second as given in the Tables III., IV., and V.

The logarithms of i and v are given for the sake of comparison.

31. *The temperature.*—The chief reason why the water from the main had been used was from the necessity of having constant temperature throughout the experiments, and my previous experience of the great constancy of the temperature of the water in the mains, even over a period of some weeks.

At the commencement of the experiments the temperature of the water when flowing freely was found to be 5 C. or 41° F., and it remained the same throughout the experiments. Nevertheless, a fact which had been overlooked caused the temperature in the pipes to vary somewhat and in a manner somewhat difficult to determine.

This fact, which was not discovered until after the experiments had been reduced, was that the temperature of the workshop being above that of the main, the water would be warmed in flowing through the pipes to an extent depending on its flow. The possibility of this had not been altogether overlooked, and an early observation was made to see if any such warming occurred, but as it was found to be less than half a degree no further notice was taken until on reducing the results it was found that the velocities obtained with the very smallest discharges presented considerable discrepancies in various experiments; this suggested the cause.

The discrepancies were not serious if explained, so that all that was necessary was to carefully repeat the experiments at the lower velocities observing the temperatures of the effluent water. This was done, and further experiments were made (see Art. 33).

Table III.

Experiments on Resistance in Pipes made January 29, 1883.

Pipe No. 4, lead.—Diameter (as measured 0·242 inch), 6·15 millims. Length: total, 16 feet; to first gauge hole, 9·6 feet; between gauge holes (5 feet), 1·524 metres. Water from the Manchester Main.

Reference number.	Pressures.			Discharges.									Temperature.		Slope of pressure in water.	Velocity in metres per second.		
	Mercury in water.	Bisulphide of carbon in water.	Reduced to metres of water.	Time in seconds taken to fill flask.				Velocity through orifice in thin plate.				Reduced to c.m. per second.	Centigrade.	Fahrenheit.	i.	v.	log i.	log v.
				1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	4. 1000 c.c.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.							
1	20	..	0·0131	130	..	..	..	..	..	..	..	2·33	12	..	0·0086	0·0785	$\bar{3}$·935	$\bar{2}$·895
2	40	..	0·0262	69	..	..	..	..	..	..	..	4·40	..	..	0·01720	0·1480	$\bar{2}$·236	$\bar{1}$·170
3	60	..	0·0393	45	..	..	..	..	..	..	..	6·73	11	..	0·0258	0·2265	$\bar{2}$·412	$\bar{1}$·355
4	80	..	0·0524	34	..	..	..	..	..	..	..	8·91	..	..	0·0345	0·3000	$\bar{2}$·537	$\bar{1}$·477
5	100	..	0·0656	28	..	..	..	..	..	..	..	10·70	10	..	0·0430	0·3640	$\bar{2}$·634	$\bar{1}$·561
6	120	..	0·0787	23	..	..	..	..	..	..	..	13·2	9	..	0·0516	0·4426	$\bar{2}$·713	$\bar{1}$·646
7	140	Unsteady	0·0918	21	..	..	..	..	..	..	..	14·5	8	..	0·0602	0·4865	$\bar{2}$·780	$\bar{1}$·687
8	160	..	0·1040	..	80	..	..	..	..	..	..	14·5	7	..	0·0682	0·5106	$\bar{2}$·834	$\bar{1}$·708
9	160 } *l*	..	0·1040	..	76	..	..	..	..	..	..	15·2	6	..	0·0682	0·5106	$\bar{2}$·834	$\bar{1}$·708
10	180	..	0·1181	..	71	..	..	..	..	..	..	16·3	..	..	0·0774	0·5483	$\bar{2}$·889	$\bar{1}$·739
11	200	..	0·1313	..	71	..	..	..	..	..	..	16·3	..	..	0·0861	0·5483	$\bar{2}$·935	$\bar{1}$·739
12	220	..	0·1443	..	69	..	..	..	..	..	..	16·8	..	..	0·0946	0·5650	$\bar{2}$·976	$\bar{1}$·752
13	240	..	0·1574	..	67	..	..	..	..	..	..	17·3	..	..	0·1033	0·5822	$\bar{1}$·014	$\bar{1}$·765
14	260	..	0·1707	..	66·5	..	..	..	..	..	..	17·4	..	..	0·1120	0·5862	$\bar{1}$·049	$\bar{1}$·768
15	280	..	0·1837	..	64	..	..	..	..	..	..	18·1	6	..	0·1206	0·6096	$\bar{1}$·081	$\bar{1}$·785
16	300	..	0·1968	..	61·5	..	..	..	..	..	..	18·8	..	..	0·1292	0·6339	$\bar{1}$·111	$\bar{1}$·802
17	320	..	0·2099	..	60	..	..	..	..	..	..	19·3	..	..	0·1378	0·6520	$\bar{1}$·139	$\bar{1}$·813

Table III. (continued).

Reference number.	Pressures.			Discharges.									Temperature.		Slope of pressure in water.	Velocity in metres per second.		
	Mercury in water.	Bisulphide of carbon in water.	Reduced to metres of water.	Time in seconds taken to fill flask.				Velocity through orifice in thin plate.				Reduced to c.m. per second.	Centigrade.	Fahrenheit.				
				1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	4. 1000 c.c.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.				i.	v.	log i.	log v.
18	320 ⎫	..	0·2099	..	..	..	..	4·7	..	..	..	19·1	..	..	0·1378	0·6413	$\bar{1}$·139	$\bar{1}$·807
19	400 ⎪	..	0·2613	..	54	..	..	5·3	..	..	..	21·5	..	..	0·1714	0·7228	$\bar{1}$·234	$\bar{1}$·859
20	500 ⎬ l	..	0·3274	..	..	..	..	6·0	..	..	..	24·3	..	..	0·2148	0·8185	$\bar{1}$·332	$\bar{1}$·913
21	700 ⎪	..	0·4592	..	..	..	..	7·4	..	..	..	30·0	..	..	0·3014	1·033	$\bar{1}$·479	0·014
22	1000 ⎭	..	0·6562	..	..	..	..	9·4	..	..	..	38·1	..	..	0·4306	1·283	$\bar{1}$·634	0·108
23	1500 ⎫	..	0·9355	..	..	..	..	11·7	..	..	..	47·5	5	..	0·6138	1·268	$\bar{1}$·788	0·103
24	2000 ⎪	..	1·2480	..	..	..	..	13·6	..	..	..	55·1	..	..	0·8185	1·854	$\bar{1}$·913	0·268
25	2500 ⎪	..	1·5560	..	..	..	..	15·8	..	..	..	64·2	..	..	1·021	2·158	0·009	0·334
26	3000 ⎪	..	1·8710	..	..	..	..	17·5	..	..	..	71·0	..	..	1·228	2·388	0·089	0·378
27	3500 ⎪	..	2·1830	..	..	..	..	19·1	..	..	..	79·1	..	..	1·433	2·661	0·156	0·425
28	4000 ⎬ r	..	2·4950	..	..	..	..	20·1	..	..	..	81·1	..	..	1·637	2·729	0·214	0·436
29	4000 ⎪	..	2·4950	..	..	..	..	..	4·9	..	..	79·5	..	..	1·637	2·674	0·214	0·427
30	5000 ⎪	..	3·1120	..	..	..	..	..	5·7	..	..	92·5	..	..	2·042	3·112	0·310	0·493
31	6000 ⎪	..	3·7420	..	..	..	..	..	6·5	..	..	105·0	..	..	2·455	3·540	0·390	0·549
32	7000 ⎪	..	4·2660	..	..	..	..	..	7·1	..	..	115·0	..	..	2·865	3·873	0·457	0·588
33	8000 ⎭	..	4·9890	..	..	..	..	..	7·7	..	..	125·0	..	..	3·274	4·198	0·515	0·623
34	8000 ⎫ l	..	5·1290	..	..	..	..	..	8·0	..	..	130·0	..	..	3·444	4·467	0·537	0·650
35	9000 ⎭	..	5·9030	..	..	..	..	..	8·6	..	..	139·0	5	..	3·873	4·689	0·588	0·671

TABLE IV.

Conditions the same as in Table III., except the temperatures at the lower velocities.

Reference number.	Pressures.			Discharges.									Temperature.		Slope of pressure in water.	Velocity in metres per second.		
	Mercury in water.	Bisulphide of carbon in water.	Reduced to metres of water.	Time in seconds taken to fill flask.				Velocity through orifice in thin plate.				Reduced to c.m. per second.	Centigrade.	Fahrenheit.				
				1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	4. 1000 c.c.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.				i.	v.	log i.	log v.
36	20	..	0·01313	..	..	227	..	..	..	..	..	1·750	10	50	0·008591	0·0740	$\bar{3}$·934	$\bar{2}$·869
37	40	..	0·02625	..	..	131	..	..	..	..	..	3·289	8	46·4	0·01718	0·1390	$\bar{2}$·235	$\bar{1}$·143
38	60	..	0·03936	..	..	80	..	..	..	..	..	4·966	7	44·6	0·02577	0·2100	$\bar{2}$·411	$\bar{1}$·322
39	80 } l	..	0·05249	..	..	61	..	..	..	..	..	6·517	6	42·8	0·03436	0·2755	$\bar{2}$·536	$\bar{1}$·440
40	100	..	0·06562	..	..	50·5	..	..	..	..	..	7·871	5	41	0·04296	0·3327	$\bar{2}$·633	$\bar{1}$·522
41	120	..	0·07871	..	..	..	86	..	..	..	..	11·650	5	41	0·05153	0·3918	$\bar{2}$·712	$\bar{1}$·593
42	140	..	0·09184	..	..	..	76	..	..	..	..	13·16	5	41	0·06296	0·4426	$\bar{2}$·779	$\bar{1}$·646
43	160	..	0·1040	..	..	..	66	..	..	..	..	15·18	5	41	0·06808	0·5106	$\bar{2}$·833	$\bar{1}$·708
44	180	Unsteady	0·1181	..	..	..	62	..	..	..	..	16 15	5	41	0·07727	0·5433	$\bar{2}$·888	$\bar{1}$·735
45	200	..	0·1313	..	..	..	61	..	..	..	..	16·41	5	41	0·08591	0·5521	$\bar{2}$·934	$\bar{1}$·742
46	220	..	0·1443	..	..	..	60	..	..	..	..	16·52	5	41	0·09441	0·5560	$\bar{2}$·975	$\bar{1}$·745
47	240 } l	..	0·1574	..	..	..	58	..	..	..	..	17·26	5	41	0·1031	0·5808	$\bar{1}$·013	$\bar{1}$·764
48	280	..	0·1837	..	..	..	55	..	..	..	..	18·20	5	41	0·1203	0·6124	$\bar{1}$·080	$\bar{1}$·787
49	320	..	0·2099	..	..	..	52	..	..	..	..	19·06	5	41	0·1375	0·6413	$\bar{1}$·138	$\bar{1}$·807
50	360	..	0·2250	..	..	..	50	..	..	..	..	20·00	5	41	0·1473	0·6730	$\bar{1}$·168	$\bar{1}$·828
51	400	..	0·2625	..	..	..	47	..	..	..	..	21·29	..	41	0·1718	0·7162	$\bar{1}$·235	$\bar{1}$·855

TABLE V.

Pipe No. 5, lead.—Diameter (as measured 0·498 inch), 12·7 millims. Length: total, 16 feet; to first gauge hole, 9·6 feet; between gauge holes (5 feet), 1·524 metres. Water from the Manchester Main.

Reference number.	Pressures. Mercury in water.	Pressures. Bisulphide of carbon in water.	Pressures. Reduced to metres of water.	Discharges. Time in seconds taken to fill flask. 1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	Velocity through orifice in thin plate. Gauge No. 1.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.	Reduced to c.m. per second.	Temperature. Centigrade.	Fahrenheit.	Slope of pressure in water. i.	Velocity in metres per second. v.	log i.	log v.
52	..	1	0·001219	..	..	..	4·5	..	..	..	..	4·346	12	..	0·00080	0·0346	$\bar{4}$·902	$\bar{2}$·539
53	..	2	0·002438	..	..	..	8·4	..	..	..	..	8·110	..	..	0·00159	0·0646	$\bar{3}$·203	$\bar{2}$·810
54	..	3	0·003656	..	..	..	11·2	..	..	..	..	9·841	..	..	0·00239	0·0784	$\bar{3}$·379	$\bar{2}$·894
55	..	4	0·004876	..	..	..	16·4	..	..	..	..	15·85	11	..	0·00319	0·1262	$\bar{3}$·504	$\bar{1}$·101
56	..	5	0·006082	..	..	..	..	4·4	..	..	..	17·83	..	..	0·00398	0·1420	$\bar{3}$·600	$\bar{1}$·152
57	..	6	0·007312	..	..	..	..	5·3	..	..	..	21·48	10	..	0·00478	0·1711	$\bar{3}$·680	$\bar{1}$·233
58	..	7 } l	0·008532	..	..	..	..	6·0	..	..	..	24·33	..	..	0·00558	0·1937	$\bar{3}$·747	$\bar{1}$·287
59	..	8	0·009750	..	..	..	..	7·0	..	..	..	28·38	8	..	0·00638	0·2260	$\bar{3}$·805	$\bar{1}$·354
60	Un-steady	9	0·01097	..	..	..	..	7·0	..	..	..	28·38	..	..	0·00717	0·2260	$\bar{3}$·856	$\bar{1}$·354
61	..	10	0·01219	..	..	..	..	7·6	..	..	..	30·84	7	..	0·00798	0·2455	$\bar{3}$·902	$\bar{1}$·390
62	..	11	0·01340	..	..	..	..	8·0	..	..	..	32·44	..	..	0·00877	0·2583	$\bar{3}$·943	$\bar{1}$·412
63	..	20 b	0·01365	..	..	..	..	8·0	..	..	..	32·44	..	..	0·00893	0·2583	$\bar{3}$·951	$\bar{1}$·412
64	..	12	0·01463	..	..	..	..	8·4	..	..	..	34·05	..	..	0·00957	0·2710	$\bar{3}$·981	$\bar{1}$·433
65	..	13 } l	0·01582	..	..	..	..	8·6	..	..	..	34·84	..	..	0·01036	0·2774	$\bar{2}$·015	$\bar{1}$·443
66	..	14	0·01707	..	..	..	..	8·8	..	..	..	35·65	..	..	0·01117	0·2838	$\bar{2}$·048	$\bar{1}$·453
67	..	15	0·01837	..	..	..	..	9·0	..	..	..	36·48	6	..	0·01203	0·2905	$\bar{2}$·080	$\bar{1}$·463

TABLE V. (continued).

Reference number.	Pressures.			Discharges.									Temperature.		Slope of pressure in water.	Velocity in metres per second.		
	Mercury in water.	Bisulphide of carbon in water.	Reduced to metres of water.	Time in seconds taken to fill flask.			Velocity through orifice in thin plate.					Reduced to c.m. per second.	Centigrade.	Fahrenheit.				
				1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	Gauge No. 1.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.				i.	v.	log i.	log v.
68	..	40	0·02729	..	..	..	..	10·8	..	..	..	43·76	..	..	0·01787	0·3484	$\bar{2}$·252	$\bar{1}$·542
69	..	60	0·04093	..	..	..	..	13·6	..	..	..	55·09	..	..	0·02680	0·4386	$\bar{2}$·428	$\bar{1}$·642
70	..	80	0·05458	..	..	..	..	16·3	..	..	..	66·07	..	..	0·03573	0·5261	$\bar{2}$·553	$\bar{1}$·721
71	..	100	0·06824	..	..	..	..	18·2	..	..	..	73·80	..	..	0·04467	0·5875	$\bar{2}$·650	$\bar{1}$·769
72	..	120	0·08185	..	..	..	..	20·2	..	..	..	81·85	..	..	0·05483	0·653	$\bar{2}$·739	$\bar{1}$·814
73	..	120	..	..	..	..	..	..	4·8	..	..	77·81	5	..	..	0·620	..	$\bar{1}$·792
74	..	140	0·9550	..	..	..	..	..	5·2	..	..	84·34	..	..	0·06252	0·672	$\bar{2}$·796	$\bar{1}$·827
75	..	160 } l	0·1092	..	..	..	..	..	5·8	..	..	93·98	..	..	0·07145	0·749	$\bar{2}$·854	$\bar{1}$·874
76	..	170	0·1159	..	..	..	..	..	6·0	..	..	97·28	..	..	0·07586	0·775	$\bar{2}$·880	$\bar{1}$·889
77	..	180	0·1228	..	..	..	..	..	6·3	..	..	102·1	..	..	0·08036	0·813	$\bar{2}$·905	$\bar{1}$·910
78	..	190	0·1295	..	..	..	..	..	6·5	..	..	105·2	..	..	0·08473	0·838	$\bar{2}$·928	$\bar{1}$·923
79	..	200	0·1365	..	..	..	..	..	6·6	..	..	107·0	..	..	0·08934	0·852	$\bar{2}$·951	$\bar{1}$·930
80	..	216	0·1433	..	..	..	..	..	6·8	..	..	110·2	..	..	0·09376	0·878	$\bar{2}$·972	$\bar{1}$·943
81	..	220	0·1500	..	..	..	..	..	7·0	..	..	113·6	..	..	0·09818	0·904	$\bar{2}$·992	$\bar{1}$·956
82	..	230	0·1567	..	..	..	..	..	7·3	..	..	118·4	..	..	0·1026	0·942	$\bar{1}$·011	$\bar{1}$·974
83	..	240	0·1637	..	..	..	..	..	7·5	..	..	121·7	..	..	0·1072	0·993	$\bar{1}$·030	$\bar{1}$·986
84	250 } l	..	0·1637	..	..	..	..	..	7·5	..	..	..	5	..	0·1072	..	$\bar{1}$·030	..
85	300	..	0·1968	..	..	..	..	..	8·6	..	..	139·4	..	..	0·1289	1·110	$\bar{1}$·110	0·045

Table V. (continued).

Reference number.	Pressures.			Discharges.									Temperature.		Slope of pressure in water.	Velocity in metres per second.		
	Mercury in water.	Bisulphide of carbon in water.	Reduced to metres of water.	Time in seconds taken to fill flask.			Velocity through orifice in thin plate.					Reduced to c.m. per second.	Centigrade.	Fahrenheit.				
				1. 303 c.c.	2. 1160 c.c.	3. 500 c.c.	Gauge No. 1.	Gauge No. 2.	Gauge No. 3.	Gauge No. 4.	Gauge No. 5.				i.	v.	log i.	log v.
86	400 } l	..	0·2625	..	..	..	..	..	10·3	..	..	166·8	..	..	0·1718	1·328	$\bar{1}$·235	0·123
87	500	..	0·3281	..	..	..	..	..	11·7	..	..	189·7	..	..	0·2148	1·511	$\bar{1}$·332	0·179
88	1,000	..	..	..	..	..	..	..	11·5	..	..	186·3	..	..	..	1·483	..	0·171
89	1,500	..	0·4798	..	..	..	..	..	14·2	..	..	230·2	..	..	0·3141	1·833	$\bar{1}$·497	0·263
90	2,000	..	0·6398	..	..	..	..	..	17·3	..	..	280·6	..	..	0·4188	2·234	$\bar{1}$·622	0·349
91	2,000	..	..	..	..	..	..	..	..	8·3	..	269·2	..	..	..	2·143	..	0·331
92	3,000	..	0·9595	..	..	..	..	..	..	10·8	..	350·0	..	..	0·6281	2·787	$\bar{1}$·798	0·445
93	5,000	..	1·596	..	..	..	..	..	..	18·7	..	479·8	..	..	1·045	3·820	0·019	0·582
94	7,000 } b	..	2·239	..	..	..	..	..	..	..	8·7	605·4	..	..	1·466	4·820	0·166	0·683
95	7,000	..	2·239	..	..	..	..	..	..	..	10·2	563·7	..	..	1·466	4·488	0·166	0·652
96	9,000	..	2·878	..	..	..	..	..	..	..	11·4	660·7	..	..	1·884	5·261	0·275	0·721
97	11,000	..	3·516	..	..	..	..	..	..	..	12·4	736·3	..	..	2·302	5·875	0·362	0·769
98	13,000	..	4·150	..	..	..	..	..	..	..	13·5	803·6	5	..	2·717	6·398	0·434	0·806
99	15,000	..	4·798	..	..	..	..	..	..	..	13·7	875·0	..	..	3·141	6·967	0·497	0·843
100	15,500	..	4·955	..	..	..	..	..	..	..	..	887·2	5	..	3·244	7·064	0·511	0·849

32. *The results of the experiments.*—A considerable number of preliminary experiments were made until the results showed a high degree of consistency. Then a complete series of experiments were made consecutively with each tube. The results of these are given in Tables III. and V.

33. *The critical velocities.*—The determination of these, which had been the main object of the experiments, was to some extent accomplished directly during the experiments, for starting from the very lowest velocities, it was found that the fluid in the differential gauge was at first very steady, lowering steadily as the velocity was increased by stages, until a certain point was reached, when there seemed to be something wrong with the gauge. The fluid jumped about, and the smallest adjustment of the tap controlling the velocity sent the fluid in the gauge out of the field of the microscope. At first this unsteadiness always came upon me as a matter of surprise, but after repeating the experiments several times, I learnt to know exactly when to expect it. The point at which this unsteadiness is noted is marked in the tables.

It was not, however, by the unsteadiness of the pressure gauge that the critical velocity was supposed to be determined, but by comparing the ratio of velocities and pressures given in the columns v and i in the tables. This comparison is shown in diagram I., Plate 74, the values of i being abscissæ and v ordinates. It is thus seen that for each tube the points which mark the experiments lie very nearly in a straight line up to definite points marked C, at which divergence sets in rapidly.

The points at which this divergence occurs correspond with the experiments numbered 6 and 59, which are immediately above those marked unsteady.

Thus the change in the law of pressure agrees with the observation of unsteadiness in fixing the critical velocities.

According to my assumption, the straightness of the curves between the origin and the critical points would depend on the constancy of temperature, and it was the small divergences observed that suggested a variation of temperature which had been overlooked. This variation was confirmed by further experiments, amongst which are those contained in Table IV. These showed that the probable variation of the temperature was in Table III. from 12° C. to 9° C. at the critical point, and from 12° C. to 8° C. in Table V., which variations would account for the small deviation from the straight.

It only remained, then, to ascertain how far the actual values of v_c, the velocity at the critical points, corresponded with the ratio $\frac{\mu}{D}$ or $\frac{P}{D}$.

For tube 4 from the Table III.

$$D=0^{m}\cdot00615$$

$$v_c=0^{m}\cdot4426$$

at 9° C.; at this temperature

$$P=\cdot757$$

see p. 952.

Hence putting

$$B_c = \frac{P}{v_c D}$$

we have

$$B_c = 279 \cdot 7$$

Again, for tube 5, Table V.

$$D = \cdot 0127$$

$$v_c = \cdot 2260$$

at 8° C.; at which temperature

$$P = \cdot 7796$$

whence

$$B_c = 272 \cdot 0$$

The differences in the values of B_c thus obtained would be accounted for by a variation of a quarter of a degree in temperature, and hence the results are well within the accuracy of the experiments.

To each critical velocity, of course, there corresponds a critical value of the pressure. These are determined as follows.

The theoretical law of resistance for steady motion may be expressed

$$A_c D^2 i = B_c P v$$

and multiplying both sides by $\frac{D}{P^2}$

$$\frac{A_c D^3 i}{P^2} = B_c \frac{D}{P} v$$

This law holds up to the critical velocity, and then the right hand number is unity if B_c has the values just determined.

$$A_c = \frac{P^2}{D^3 i_c}$$

by Table III.

$$i_c = \cdot 0516$$

$$P^2 = \cdot 573$$

$$D^3 = \cdot 000{,}000{,}232$$

which give

$$A_c = 47{,}750{,}000$$

By Table V.

$$i_c = \cdot 00638$$

$$P^2 = \cdot 607$$

$$D^3 = \cdot 00000205$$

which give

$$A_c = 46{,}460{,}000$$

which values of A_c differ by less than by what would be caused by half a degree of temperature.

The conclusion, therefore, that the critical velocity would vary as

$$\frac{\mu}{D}$$

is abundantly verified.

34. *Comparison with the discharges calculated by* POISEUILLE'S *formula.*—POISEUILLE experimented on capillary tubes of glass between ·02 and ·1 millim. in diameter, and it is a matter of no small interest to find that the formula of discharges which he obtained from these experiments is numerically exact for the bright metal tubes 100 times as large.

POISEUILLE'S formula is—

$$Q=1836{\cdot}724(1+0{\cdot}0336793\,T+0{\cdot}000220992\,T^2)\frac{HD^4}{L}$$

T = temperature in degrees centigrade.
H = pressure in millims. mercury.
D = diameter in millims.
L = length in millims.
Q = discharge in millims. cubed.

Putting

$$i=\frac{13{\cdot}64H}{L}$$

$$P=1+(0{\cdot}336793\,T+0{\cdot}000220992\,T^2)^{-1}$$

$$v=\frac{4Q}{\pi D^2}$$

and changing the units to metres and cubic metres this formula may be written

$$47700000\frac{D^3}{P^2}i=278\frac{D}{P}v$$

the coefficients corresponding to A_c and B_c.

The agreement of this formula with the experimental results from tubes 4 and 5 is at once evident. The actual and calculated discharges differ by less than 2 per cent., a difference which would be more than accounted for by an error of half a degree in the temperature.

35. *Beyond the critical point.*—The tables show that, beyond the critical point, the relation between i and v differs greatly from that of a constant ratio; but what the exact relation is, and how far it corresponds in the two tubes, is not to be directly seen from the tables.

In the curves (Plate 74, diagram I.) which result from plotting i and v, it appears that after a period of flatness the curves round off into a parabolic form; but whether they are exact parabolæ, or how far the two curves are similar with different parameters, is difficult to ascertain by any actual comparison of the curves themselves, which, if plotted to a scale which will render the small differences of pressure visible, must extend 10 feet at least.

36. *The logarithmic method.*—So far the comparison of the results has been effected by the natural numbers, but a far more general and clearer comparison is effected by treating the logarithms of l and v.

This method of treating such experimental results was introduced in my paper on Thermal Transpiration (see Phil. Trans., Part II., 1879, p. 753).

Instead of curves, of which i and v are the abscissæ and ordinates, log i and log v are taken for the abscissæ and ordinates, and the curve so obtained is the logarithmic homologue of the natural curve.

The advantage of the logarithmic homologues is that the *shape* of the curve is made independent of any constant parameters, such parameters affecting the position of all points on the logarithmic homologue similarly. Any similarities in shape in the natural curves become identities in shape in the logarithmic homologues. How admirably adapted these logarithmic homologues are for the purpose in hand is at once seen from diagram II., Plate 73, which contains the logarithmic homologues of the curves for both pipes 4 and 5.

A glance shows the similarity of these curves, and also their general character. But it is by tracing one of the curves, and shifting the paper rectangularly until the traced curve is superimposed on the other, that the exact similarity is brought out. It appears that, without turning the paper at all, the two curves almost absolutely fit.

It also appears that the horizontal and vertical components of the shift are—

Horizontal shift	·913
Vertical shift	·294

which are within the accuracy of the work respectively identical with the differences of the logarithms of $\frac{D^3}{P^2}$ and $\frac{D}{P}$ for the two tubes.

37. *The general law of resistance in pipes.*—The agreement of the logarithmic homologues shows that not only at the critical velocities but for all velocities in these two pipes, pressure which renders $\frac{D^3}{\mu^2}i$ the same in both pipes corresponds to velocities which render $\frac{D}{\mu}v$ the same in both pipes. This may be expressed in several ways. Thus if the tabular value of i for each pipe plotted in a scale be multiplied by a number proportional to $\frac{D^3}{P^2}$ for that particular pipe and the values of v by a number proportional to $\frac{D}{P}$, then the curves which have these reduced values of i and v for abscissæ and ordinates will be identical.

A still more general expression is that if

$$i = F(v)$$

expresses the relation between i and v for a pipe in which $D = 1$, $T=0$, $P=1$.

$$\frac{D^3 i}{P^2} = F\left(\frac{Dv}{P}\right)$$

expresses the relation for every pipe and every condition of the water.

The determination of the relation between circumstances of motion and the physical condition of the water in such a general form was not contemplated when the experiments were undertaken, and must be considered as a result of the method of logarithmic homologues which brought out the relation in such a marked manner that it could not be overlooked. Nor is this all.

It had formed no part of my original intention to re-investigate the law of resistance in pipes for velocities above the critical value, as this is ground which had been very much experimented upon, and experiments seemed to show that the law was either indefinite or very complex—a conclusion which did not seem inconsistent with the supposition that above this point the resistance depended upon eddies which might be somewhat uncertain in their action. But although it was not my intention to investigate laws, I had made a point of continuing the experiments through a range of pressures and velocities very much greater I think than had ever been attempted in the same pipe.

Thus it will be noticed that in the larger tube the pressure in the last experiment is four thousand times as large as in the first. In choosing the great range of pressures I wished to bring out what previous experiments had led me to expect, namely, that in the same tube for sufficiently small pressures the pressure is proportional to the velocity, and for sufficiently great pressures, the pressure was proportional to the square of the velocity. Had this been the case not only would the lowest portion of the logarithmic homologues up to the critical point have come out straight lines inclined at 45 degrees, but the final portion of the curve would have come out a straight line at half this inclination, or with a slope of two horizontal to one vertical.

The near approach of the lower portions of the curve to the line at 45° led me, as I have already explained, to discover that the temperatures had risen at the lower velocities, and to make a fresh set of experiments, some of which are given in Table IV., in which, although the temperatures were not constant, they were sufficiently different from the previous ones to show that the discrepancy in the lower portions of the curves might be attributed to variations of temperature, and the agreement with the line of 45° considered as within the limits of accuracy of experiment.

When the logarithms of the upper portions of the curve came to be plotted, the straightness and parallelism of the two lines was very striking.

There are a few discrepancies which could not be in any way attributed to temperature, as with so much water moving this was very constant, but on examination it was seen that these discrepancies marked the changes of the discharge gauges. The law of flow through the orifices not having been strictly as the square roots of the heights, the manner in which the gauges had been compared forbade the possibility of there being a general error from this cause; the middle readings on the gauge were correct, so that the discrepancies, which are small, are mere local errors.

This left it clear that whatever might be their inclination the lines expressed the laws of pressures and velocities in both tubes, and since the lines are strictly parallel, this law was independent of the diameter of the tube. This point has been very carefully examined, for it is found that the inclination of these lines differs decidedly from that of 2 to 1, being 1·723 to 1, and so giving a law of pressures through a range 1 to 50 of

$$i \propto v^{1\cdot 723}$$

This is different from the law propounded by any of the previous experimenters, who have adhered to the laws

$$i = v^2$$

or

$$i = \mathrm{A}v + \mathrm{B}v^2$$

That neither of these laws would answer in case of the present experiments was definitely shown, for the first of these would have a logarithmic homologue inclined at 2 to 1, and the second would have a curved line. A straight logarithmic homologue inclined at a slope 1·723 to 1 means no other law than

$$i \propto v^{1\cdot 723}$$

I have therefore been at some pains to express the law deduced from my experiments on the uniform pipes so that it may be convenient for application. This law as already expressed is simply

$$\frac{\mathrm{D}^3}{\mathrm{P}^2} i = f\left(\frac{\mathrm{D}v}{\mathrm{P}}\right)$$

where f is such that

$$x = f(y)$$

is the equation to the curve which would result from plotting the resistance and velocities in a pipe of diameter 1 at a temperature zero.

The exact form of f is complex, this complexity is however confined to the region immediately after the critical point is passed.

Up to the critical point

$$\mathrm{A}_c \frac{\mathrm{D}^3}{\mathrm{P}^2} i = \mathrm{B}_c \frac{\mathrm{D}v}{\mathrm{P}}$$

After the critical point is passed the law is complex until a velocity which is 1·325 v_c is reached. Then as shown in the homologues the curve assumes a simple character again

$$\mathrm{A} \frac{\mathrm{D}^3}{\mathrm{P}^2} i = \left(\mathrm{B} \frac{\mathrm{D}v}{\mathrm{P}}\right)^{1\cdot 723}$$

that is, the logarithmic homologue becomes a straight line inclined at 1·723 to 1.

Referring to the logarithmic homologues (Plate 73, diagram II.), it will be seen that although the directions of the two straight extremities of the curve do not meet in the critical point, their intersection is at a constant distance from this point which in the logarithmic curves is, both for ordinates and abscissæ,

$$0{\cdot}154$$

These points o are therefore given by

$$\log\frac{D^3 i_c}{P^2}=\log\frac{D^3 i_o}{P^2}+0{\cdot}154$$

$$\log\frac{D v_c}{P}=\log\frac{D v_o}{P}+0{\cdot}154$$

Therefore putting

$$A=\frac{P^2}{D^3 i_o},\ B=\frac{P}{D v_o}$$

$$\log A=\log A_c+0{\cdot}154$$

$$\log B=\log B_c+0{\cdot}154$$

and by the values of A_c and B_c previously ascertained (Art. 33, p. 971),

$$\log A=8{\cdot}8311 \qquad A=67{,}700{,}000$$

$$\log B=2{\cdot}598 \qquad B=\quad 396{\cdot}3$$

We thus have for the equation to the curves corresponding to the upper straight branches

$$A\frac{D^3}{P^2}i=\left(B\,\frac{Dv}{P}\right)^{1{\cdot}723}$$

And if n have the value 1 or 1·722 according as either member of this equation is $<$ or $>$ 1 the equation

$$A\frac{D^3}{P^2}.i=\left(\frac{BDv}{P}\right)^n$$

is the equation to a curve which has for its logarithmic homologue the two straight branches intersecting in o, and hence gives the law of pressures and velocities, except those relating to velocities in the neighbourhood of the critical point, and these are seldom come across in practice.

By expressing n as a discontinuous function of $B_c\frac{Dv}{P}$ the equation may be made to fit the curve throughout.

38. *The effect of temperature.*—It should be noticed that although the range is comparatively small, still the displacement of the critical point in Tables III. and IV. is distinctly marked. The temperatures were respectively 9° C., 5° C.

$$\text{At } 9^\circ \log P^{-1}=0{\cdot}12093$$

$$\text{At } 5^\circ \log P^{-1}=0{\cdot}06963$$

$$\text{Difference}\quad ={\cdot}05130$$

This should be the differences in the values of log v_c in Tables III. and IV. The actual difference is ·062. Also the differences in log i_c should be the differences in P^2 or ·10260, whereas the actual difference is ·121.

The errors correspond to a difference of about 1° C., which is a very probable error.

It would be desirable to make experiments at higher temperature, but there were great difficulties about this which caused me, at all events for the time, to defer the attempt.

SECTION IV.

Application to DARCY'S *experiments.*

39. DARCY'S *experiments.*—The law of resistance came out so definitely from my experiments that, although beyond my original intention, I felt constrained to examine such evidence as could be obtained of the actual experimental results obtained by previous experimenters.

The lower velocities, up to the critical value, were found, as has already been shown (Art. 35), to agree exactly with POISEUILLE'S formula.

For velocities above the critical values the most important experiments were those of DARCY—approved by the Academy of Sciences and published 1845—on which the formula in general use has been founded. Notwithstanding that the formula as propounded by DARCY himself could not by any possibility fit the results which I have obtained, it seemed possible that the experiments on which he had based his law might fit my law. A comparison was therefore undertaken.

This was comparatively easy, as DARCY'S experimental results have been published in detail.

Altogether he experimented on some 22 pipes, varying in diameter from about the size of my largest, $0^m·0014$ up to $0^m·5$. They were treated in several sets, according to the material of which they were composed—wrought iron gas pipes, lead pipes, varnished iron pipes, glass pipes, new cast iron and old rusty pipes.

The method of experimenting did not differ from mine except in scale, the distance between DARCY'S gauge points being 50^m instead of 5 feet in my case. The great length between DARCY'S gauge points entailed his having joints in his pipes between these points, and the nature of his pipes was such as to preclude the possibility of a very uniform diameter. His experiments appear to have been made with extreme care and very faithfully recorded, but the irregularity in the diameters, which appears to have been as much as 10 per cent., and the further irregularity of the joints, preclude the possibility of the results of his experiments following very closely the law for uniform pipes. Another important matter to which DARCY appears to have paid but little attention was temperature. It is true that in many instances he has given the temperature, but he does not appear to have taken any account of it in his discussion of his results, although it varied as much as 20° C. in the cases where he has given it, and as his pipes, 300 metres long, were in the open air, the effect of the sun on the pipes would have led to still larger differences.

The effect of these various causes on his results may be seen, as he took the precaution to use two pressure gauges on separate lengths of 50^m of his pipes, and the records from these two gauges by no means always agree, particularly for the lower velocities. In one case the results are as wide apart as 15 and 7, and often 10 or 15 per cent. In arriving at tabular values for i he has taken the mean of the two gauges.

Taking these things into account, I could not possibly expect any close agreement with my results; still, as experiments on pipes of such large diameters are not likely to be repeated, at any rate with anything like the same care and success, they offered the only chance of proving that my law was general.

40. *Reduction of the experimental results.*—Rejecting all the experiments on rusty and rough pipes, *i.e.*, selecting the lead, the varnished, the glass, and new cast iron pipes, which ranged from half-an-inch to twenty inches diameter, I had the logarithmic homologues drawn. These are shown on diagram III., Plate 74. In the case of two of the smaller pipes the smallest velocity is well below the critical point, and in several of the other pipes the smallest velocity is near the critical velocity. This accounts for the lower ends of the logarithmic curves being somewhat twisted; for the remainder of the logarithmic homologues are nearly straight; some are slightly bent one way and some another, but they are none of them more bent than may be attributed to experimental inaccuracy.

The inclinations of the upper ends of the lead and bituminous pipes is 1·746, slightly greater than mine; but in the cases of the glass pipes and the cast iron pipes the slopes are 1·82 and 1·92 respectively.

So much appeared from the logarithmic homologues themselves, but the most important question was, would the curves agree with the results calculated from the formula

$$A\frac{D^3}{P^2}i=\left(B\frac{D}{P}v\right)^n$$

41. *Comparison with the law of resistance.*—In applying this test I was at first somewhat at a loss on account in some cases of the want of any record of the temperature, and the doubt as to such temperatures as had been recorded being the temperature of the water in the pipes between the gauges.

The dates at which the experiments were made to a certain extent supplied the deficiency of temperature, the temperatures given fixing the law of temperature, so that the probable temperature could be assumed where it was not given.

Assuming the temperature, the values of

$$i_o=\frac{P^2}{AD^3}$$

$$v_o=\frac{P}{BD}$$

were calculated for each tube, using the values of A and B as already determined, log i_o and v_0 are the co-ordinates of O the intersection of the two straight branches of the logarithmic curves, so that the application of the formula to the results was simply tested by continuing the straight upper branches of the logarithmic homologues to see whether they passed through the corresponding point O.

The agreement, which is shown in diagram III., Plate 74, is remarkable. There are some discrepancies, but nothing which may not be explained by inaccuracies, particularly inaccuracies of temperature.

42. *The effect of the temperature above the critical point.*—It is a fact of striking significance, physical as well as practical, that while the temperature of the fluid has such an effect at the lower velocities that, *cœteris paribus*, the discharge will be double at 45° C. what it is at 5° C., so little is the effect at the higher velocities that neither DARCY or any other experimenter seems to have perceived any effect at all.

In my experiments the temperature was constant, 5° C. at the higher velocities, so that I had no cause to raise this point till I came to DARCY's result, and then, after perplexing myself considerably to make out what the temperatures were, I noticed the effect of the temperature is to shift the curves 2 horizontal to 1 vertical, which corresponds with a slope of 2 to 1, and so nearly corresponds with the direction of the curves at higher velocities that variations of 5° or 10° C. produce no sensible effect; or, in other words, the law of resistance at the higher velocity is sensibly independent of the temperature, *i.e.*, of the viscosity.

Thus not only does the critical point, the velocity at which eddies come in, diminish with the viscosity, but the resistance after the eddies are established is nearly, if not quite, independent of the viscosity.

43. *The inclinations of the logarithmic curves.*—Although the general agreement of the logarithmic homologues completely establishes the relations between the diameters of the pipes, the pressures and velocities for each of the four classes of pipes tried, viz., the lead, the varnished pipes, the glass pipes, and the cast iron, there are certain differences in the laws connecting the pressures and velocity in the pipes of different material. In the logarithmic curves this is very clearly shown as a slight but definite difference between the inclination of the logarithmic homologues for the higher velocities.

The variety of the pipes tried reduces the possible causes of this difference to a small compass. It cannot be due to any difference in diameters, as at least three pipes of widely different diameters belong to each slope. It is not due to temperature. This reduces the cause for the different values of n to the irregularity in the pipes owing to joints and other causes, and the nature of the surfaces.

The effect of the joints on the values of n seems to be proved by the fact that DARCY's three lead pipes gave slightly different values for n, while my two pipes without joints gave exactly the same value, which is slightly less than that obtained from DARCY's experiments.

DARCY'S pipes were all of them uneven between the gauge points, the glass and the iron varying as much as 20 per cent. in section. The lead were by far the most uniform, so that it is not impossible that the differences in the values of n may be due to this unevenness.

But the number of joints and unevenness of the tarred pipes corresponded very nearly with the new cast iron, and between these there is a very decided difference in the value of n. This must be attributed to the roughness of the cast iron surface.

44. *Description of Diagram* III.

Diagram III.—In this diagram the experiments of POISEUILLE and DARCY are brought into comparison with those of the present investigation.

In consequence of the number of lines, the general aspect of the diagram is somewhat confused, but such confusion vanishes so soon as it is clearly perceived that each line of dots indicates the logarithmic homologue for some particular pipe as determined by experiment, reduced and plotted in exactly the same manner as for diagram II.; DD and EE being exact repetitions of the logarithmic homologue for pipes 4 and 5, on a somewhat smaller scale.

It is at once apparent from diagram III. how, for the most part, the experiments have been well below or well above the critical values. In the small pipes of POISEUILLE the velocities were below the critical values, and hence lie in straight lines inclined at 45°.

The smallest pipe on which POISEUILLE'S experimented had a diameter of 0·014 millim.; only one experiment, marked A, is shown in the diagram, as the remaining three extended outside the range of the plate. They fall exactly on the dotted line through A, and do not reach the critical value.

The same is true of all the rest of POISEUILLE'S experiments except those made on a much larger pipe, diameter 0·65 millim., hence it is thought sufficient to plot only one, namely BB.

CC shows the experimental results obtained with the pipe 0·65 millim. diameter, and these reach the critical value as given by the formula, and then diverge from the line.

It is important to notice, however, that the points are not taken directly from POISEUILLE'S experiments, which have been subjected to a correction rendered necessary by the fact that POISEUILLE did not measure the resistance by ascertaining the pressure at two points in the pipe, but by ascertaining the pressure in the vessels from which and into which the water flowed through the pipe, so that his resistance includes, besides the resistance of the pipe, the pressure necessary to impart the initial velocity to the water. This fact, which appears to have been entirely overlooked, had a very important influence on many of POISEUILLE'S results. POISEUILLE endeavoured to ascertain what was the limit to the application of his law, and, with the exception of his smallest tubes, succeeded in attaining velocities at which the results were no longer in accordance with his law.

When I first examined his experiments I expected to find these limiting velocities above the critical velocities as given by my formula. In all cases, however, they were very much below, and it was then I came to see that POISEUILLE had taken no account of the pressure necessary to start the fluid.

It then became interesting to see how far the deviations were to be explained in this way.

In pipes of sensible size the pressure necessary to start the fluid lies between

$$\frac{v^2}{2g} \text{ and } 1\cdot505\,\frac{v^2}{2g}$$

according to whether the mouthpiece is trumpet-shaped or cylindrical. POISEUILLE states that he was careful to keep both ends of his pipe cylindrical, hence according to the law for mouthpieces of sensible size, the pressures which he gives should be corrected by $1\cdot505\,\frac{v^2}{2g}$.

This correction was made, and it was then found that with all the smaller tubes POISEUILLE's law held throughout his experiments, and with the larger pipe it held up to the critical value and then diverged in exact accordance with my formula, as shown by the line CC.

DARCY's experiments in the case of three tubes F, G, I fall below the critical value, and in all these cases agree very well with the theoretical curve as regards both branches.

This, however, must be looked upon as accidental, as at the lower velocities DARCY had clearly reached the limit of sensitiveness of his pressure gauges; thus, for instance, the experiment close by the letter F is the mean of two readings which are respectively 7 and 15; there is a tendency throughout the entire experiments to irregularity in the lower readings which may be attributed to the same cause, and this seems to explain the somewhat common deviation of the one or two lower experiments from the line given by the middle dots.

A somewhat similar cause will explain cases of deviation in the one or two upper experiments, for the discrepancy in the two gauges here again becomes considerable.

For these reasons the intermediate experiments were chiefly considered in determining the slopes of the theoretical lines.

These slopes were obtained as the mean of each class of tubes :—

Lead jointed	1·79
Varnished	1·82
Glass	1·79
New cast iron	1·88
Incrusted pipe	2·
Cleaned pipe	1·91

and then in the cases in which the temperature was given, I, J, L, M, N, the points O having being determined by the formula,

$$\text{Log } i_0 = 2 \log P - 3 \log D - 7{\cdot}851$$
$$\text{Log } v_0 = \log P - \log D - 2{\cdot}598$$

the lines having the respective slopes were drawn through these points and in all cases agreed closely with the experiments.

In the cases where the temperature was not given the values of log i_0 and log v_0 were calculated for 5° C., these are shown along the line marked "line of intersections at 5°," through these points lines are shown drawn at an inclination of 2 to 1, which are the lines on which O would lie whatever might be the temperature. These with the respective slope lines were drawn so as most nearly to agree with the experiments, these intersect the lines at 2 to 1 in the points O which indicate the temperatures, and considering the extremely small effect of the temperature these are all very probable temperatures with the exception of G, H, and S, in which cases O is above the line for 5° C. This indicates strongly that in these cases there must have been a small error, 2 or 3 per cent., in determining the effective diameter of the pipes.

It seemed very probable that roughness in the pipes, such as might arise from incrustation or badly formed joints, would affect the logarithmic homologues, and for this reason only the smoother classes of pipes were treated; but with a view to test this idea, the homologues Q and R, which related to the same incrusted pipe before and after being cleaned were drawn, and their agreement is such as to show that for such pipe the effect of incrustation is confined to the effect on the diameter of the pipe, and on the value of n which it raises to 2. This, however, was a large pipe and the velocities a long way above the critical velocity, so that it is quite possible that the same incrustation in a smaller pipe would have produced a somewhat different effect.

The general result of this diagram is to show that throughout the entire range—from pipes of $0^{\cdot m}000014$ to $0^{\cdot m}5$ in diameter, and from slopes of pressure ranging from 1 to 700,000—there is not a difference of more than 10 per cent. in the experimental and calculated velocities, and with very few exceptions the agreement is within 2 or 3 per cent., and it does not appear that there is any systematic deviation whatever.

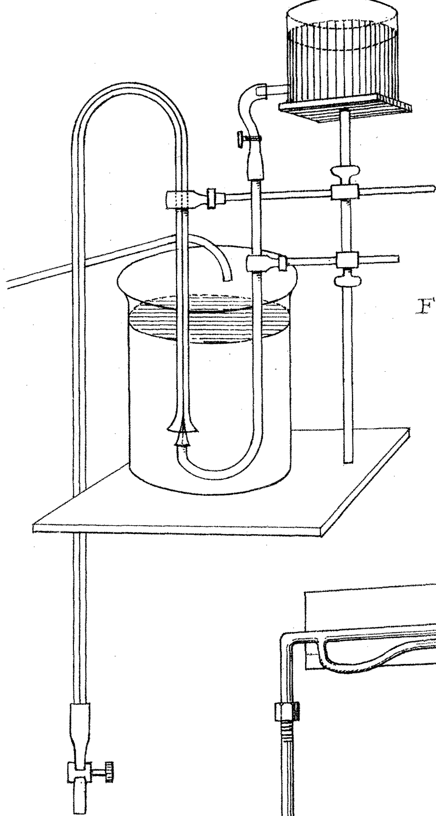


Fig. 10.

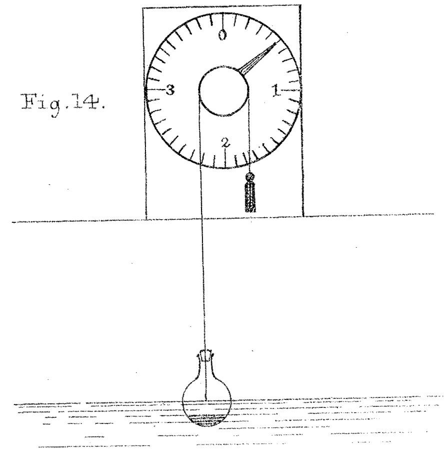


Fig. 14.

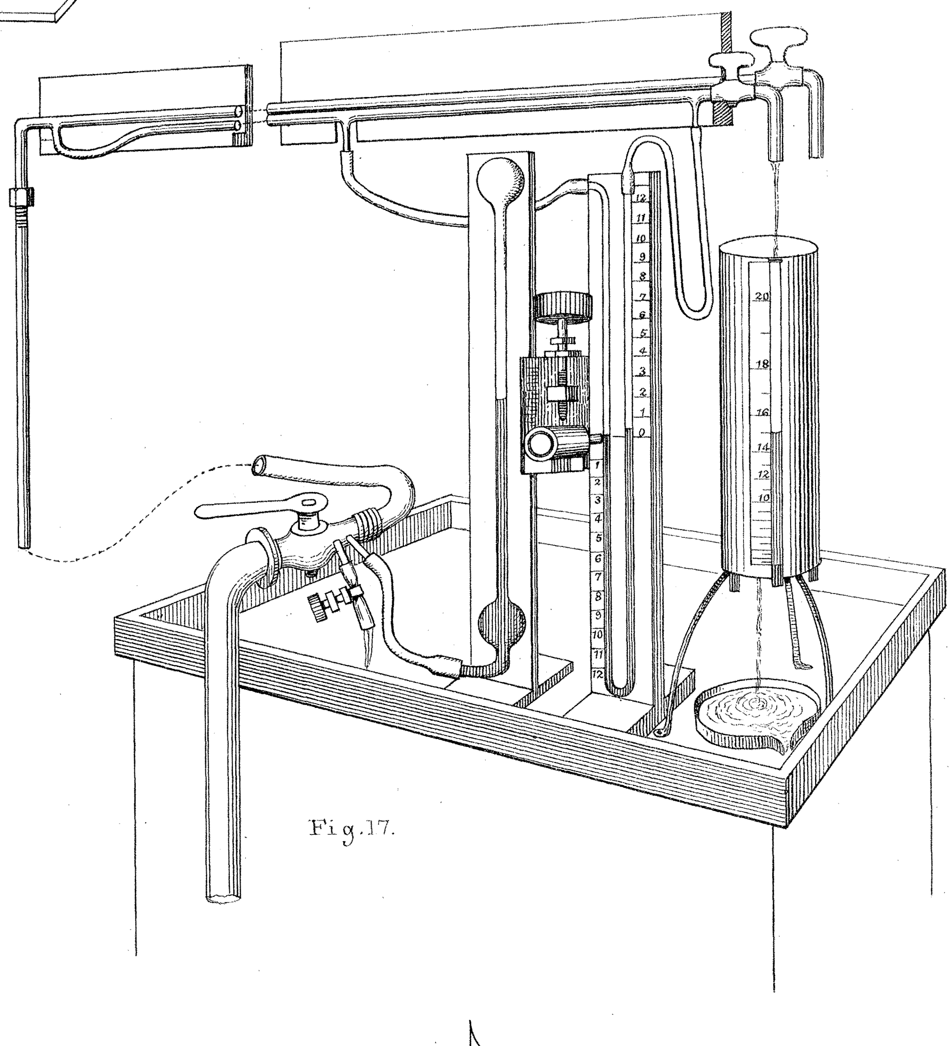


Fig. 17.

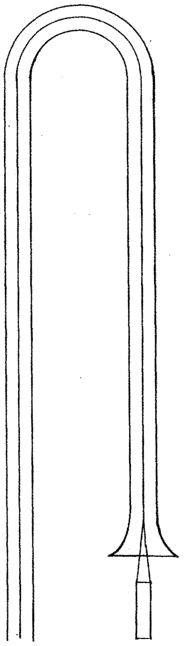


Fig. 10'.

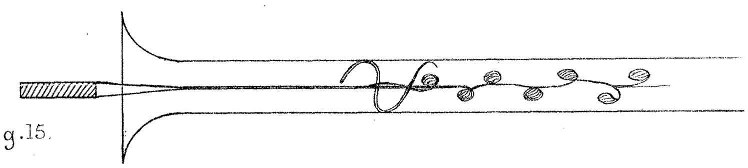


Fig. 15.

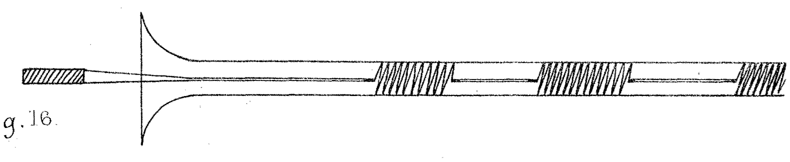


Fig. 16.

West Newman & Co. lith.

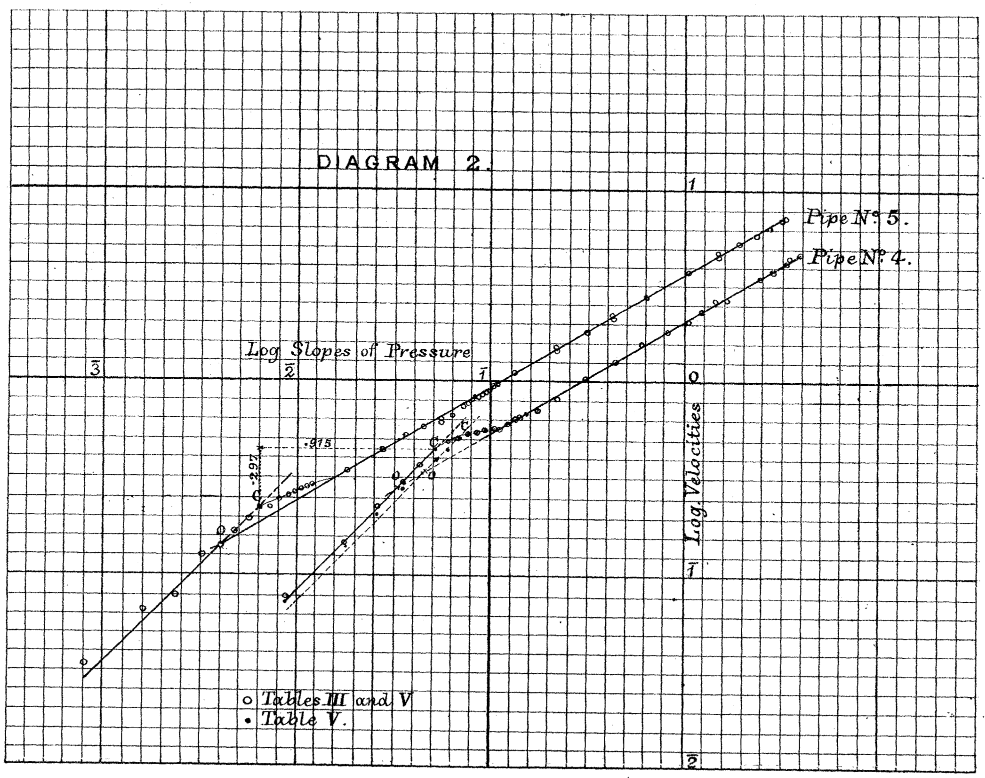


Fig. 11.

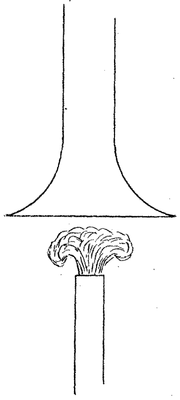


Fig. 12.

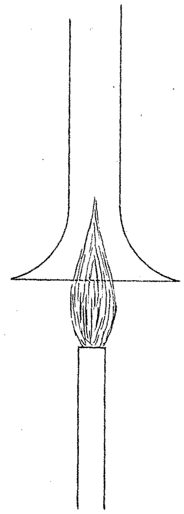
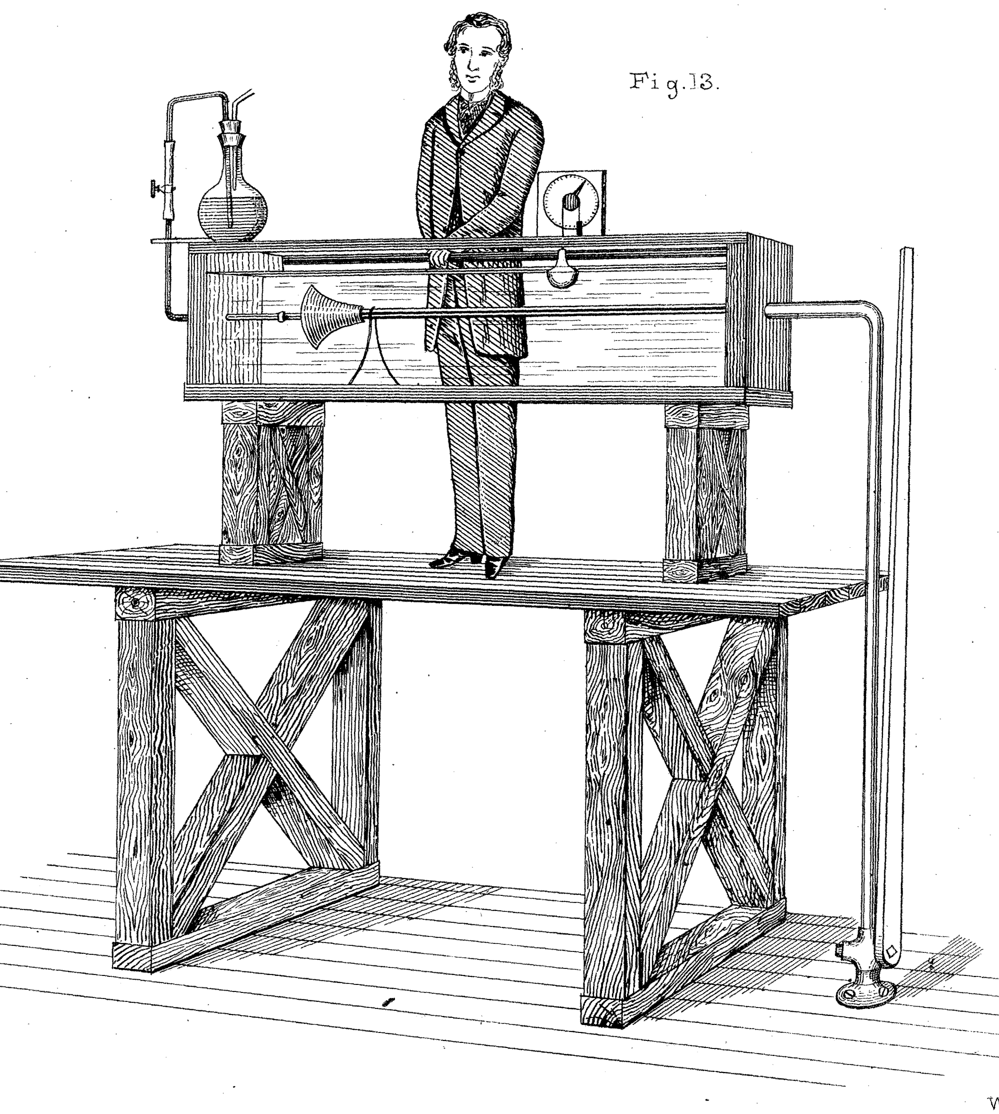
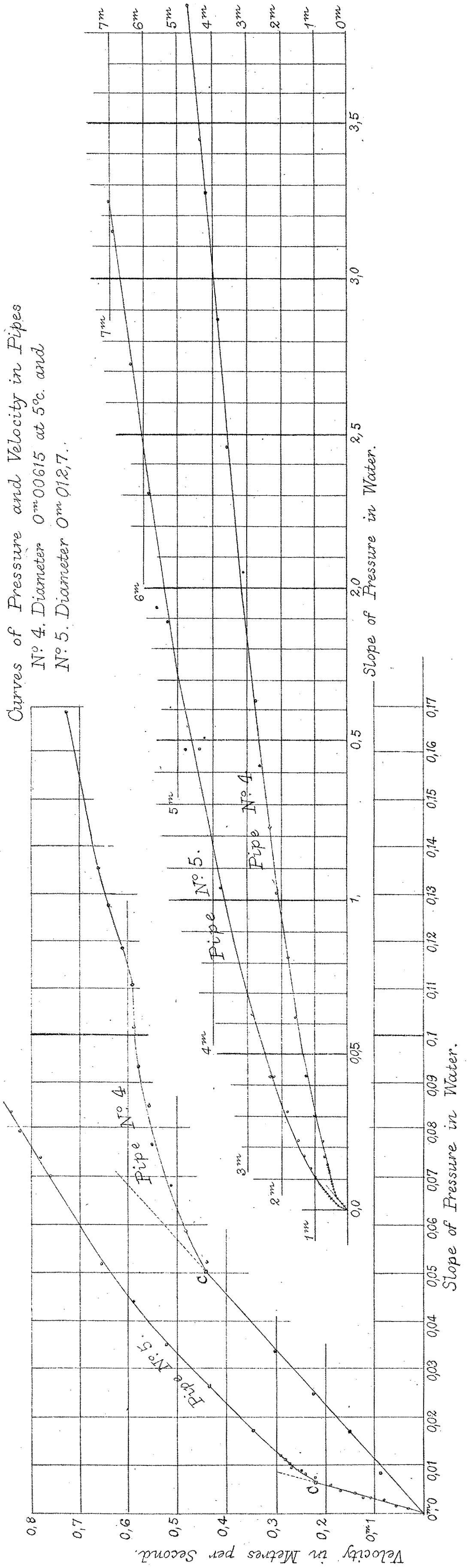
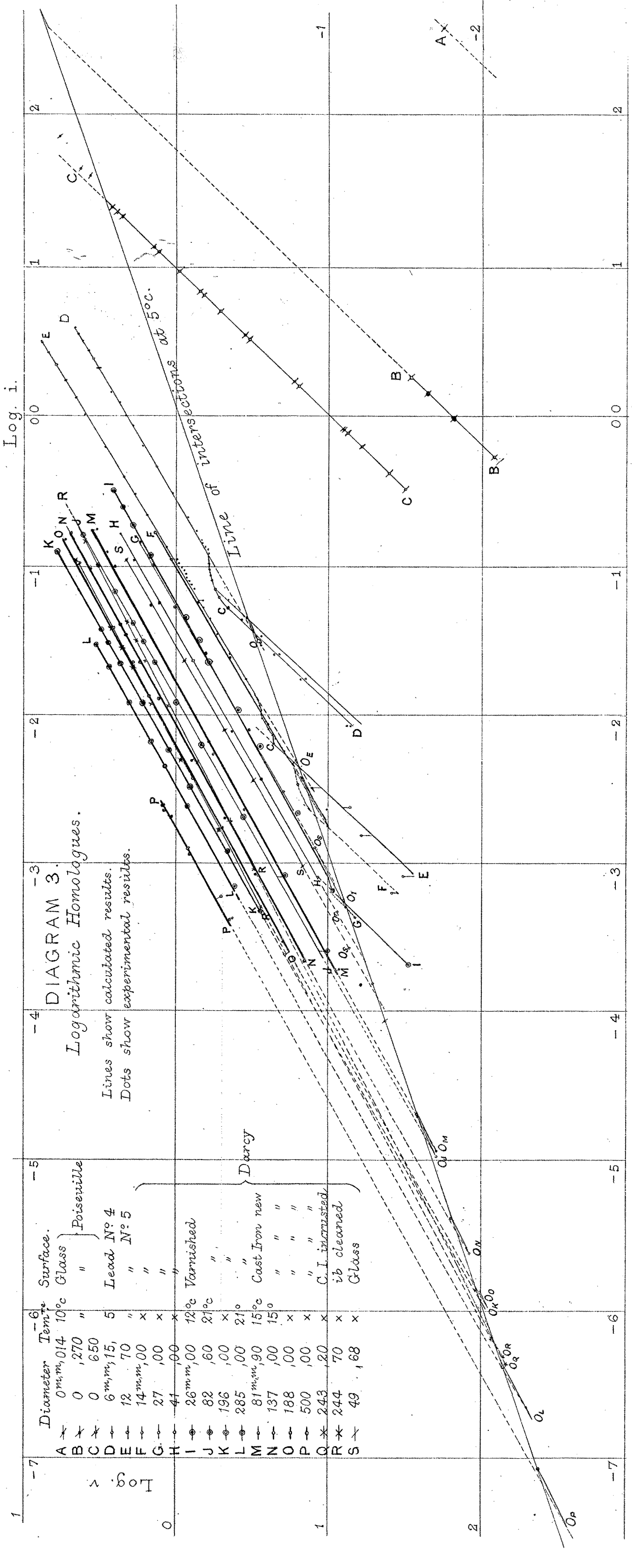


Fig. 13.







West Newman & Co. lith.